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NONLINEAR WAVE MOTION GOVERNED BY THE MODIFIED BURGERS EQUATION

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CONTENTS

The modified Burgers equation (MBE)

$$\frac{\partial V}{\partial X}+V^2\frac{\partial V}{\partial \tau}=\epsilon\frac{\partial^2 V}{\partial \tau^2}$$

has recently been shown by a number of authors to govern the evolution, with range X, of weakly nonlinear, weakly dissipative transverse waves in several distinct physical contexts. The only known solutions to the MBE correspond to the steady shock wave (analogous to the well-known Taylor shock wave in a thermoviscous fluid) or to a similarity form. It can, moreover, be proved that there can exist no Bäcklund transformation of the MBE onto itself or onto any other parabolic equation, and in particular, therefore, that no linearizing transformation of Cole–Hopf type can exist. Attempts to understand the physics underlying the MBE must then, for the moment, rest on asymptotic studies and direct numerical computation.

Our aim in the present paper is to find asymptotic solutions to the MBE for small values of the dissipation coefficient ϵ, but covering all values of the range and phase variables (X, τ) of interest, this being achieved by systematic use of matched asymptotic expansion techniques. Two specific initial distributions are studied, in which $V(0, \tau)$ is either an N-wave or a sinusoid. The corresponding problems for the ordinary Burgers equation, generalized to include cylindrical or spherical spreading effects, were treated in detail by this method elsewhere.

A feature of particular interest in the present paper is that the steady Taylor-type shock waves, which separate the lossless portions of the wave form in the early stages after shock formation, develop an internal singularity at and beyond some finite range X_1 ($X_1 = 10$ for N-waves, $X_1 = 9.601\ldots$ for the sinusoid) unless the condition $H+2G = 0$ for $X \geqslant X_1$ is imposed on the values $H(X)$, $G(X)$ of V on either side of the shock. This leads to a 'refraction' of the characteristics of the lossless solution as they pass through a shock at ranges greater than X_1, and to multiple refractions in the periodic problem.

The structure of the shock waves is analysed at all ranges, and it is shown how, at large ranges ($X = O(\epsilon^{-2})$ for N-waves, $X = O(\epsilon^{-1})$ for the sinusoid) the shocks thicken and merge with the lossless portions, leading to a phase of the motion governed by the *full* MBE. This phase is followed at still larger ranges by transition into old-age decay under linear mechanisms, and the form of the old-age functions is given. Computations of the wave form for the sinusoidal initial distribution are given that support the imposition of the criterion $H+2G = 0$ for $X > X_1$. Appendix A gives a brief derivation of the MBE for the case of transverse electromagnetic waves in a nonlinear dielectric, and Appendix B provides a sketch of some interesting features and unresolved difficulties associated with higher-order calculations.

1. Introduction

There are many fields of physics, and more particularly of continuum mechanics, in which dissipation is a significant aspect of wave propagation. Such a field is that of nonlinear acoustics (with the term extended to cover both longitudinal and transverse waves in solids). Here, in contrast to nonlinear dispersive wave theory, which has advanced greatly through the discovery, by Gardner, Greene, Kruskal and Miura, of the Inverse Scattering Transform for the exact solution of the Korteweg–de Vries equation, the evolution equations are not exactly integrable and it is not appropriate to regard the non-integrable ingredients as merely causing a slow distortion or decay of a localized solution to the exactly integrable part. For dissipative waves the essential aspect is the production, from smooth initial data, of thin *shocks* separating different signal levels, whereas with dispersion localized solitons are produced, outside which the medium is undisturbed.

There is a rich class of model equations describing the weakly nonlinear, weakly dissipative evolution of waves of acoustic type when such features as relaxation of internal degrees of freedom or tube-wall attenuation are taken into account in addition to the thermoviscous dissipation that dominates the high-frequency behaviour of ordinary acoustic media. An outline of the features of some members of this class is given by Crighton (1979). In all the cases considered there, however, the basic (weak) nonlinearity was quadratic, as is obviously required at lowest order for longitudinal waves in isotropic elastic media, whether fluids or solids. The present paper deals with a model equation for which the lowest-order nonlinearity is cubic, and for which the prototypical physical problem may be taken as involving transverse (shear) waves in an isotropic solid. Many processes are available in different media to resist the nonlinear deformation of a finite amplitude shear wave. Our assumption is here that the dominant mechanism has the nature of a viscosity, leading to a second derivative in the evolution equation for a unidirectional wave, which would then be of the form

$$\frac{\partial V}{\partial X} + V^2 \frac{\partial V}{\partial \tau} = \epsilon \frac{\partial^2 V}{\partial \tau^2}, \tag{1.1}$$

in which all variables are dimensionless, X is a range variable and τ a retarded time, or linear phase variable. Certain representations of viscoelastic solid behaviour do indeed lead to an equation of the form (1.1), as was first shown by Nariboli & Lin (1973) and subsequently by Teymur & Suhubi (1978). Nariboli & Lin also show the applicability of (1.1) to the problem of magnetohydrodynamic 'switch-on' shock waves, whereas a generalized version of (1.1), including linear terms representing dispersion and ray-tube area change, was shown by Gorschkov *et al.* (1974) to describe the electric field in a nonlinear isotropic dielectric (V then referring either to the axial electric field or the azimuthal magnetic field in a cylindrically diverging electromagnetic field). Equation (1.1) has more recently arisen in the discussion by Sugimoto *et al.* (1982) of torsional waves in a thin viscoelastic rod. Because this model equation is much less familiar than the customary Burgers equation and its generalizations on the linear side, a brief derivation of it was thought to be in order here, and is given in Appendix A for the simplest case, that of electromagnetic waves in a nonlinear dielectric. It is, in any event, clear from the physical situations referred to above that (1.1) is a canonical nonlinear evolution equation describing *transverse* waves in an isotropic dissipative medium, and that the nature of its solutions is of fundamental interest in wave theory.

There are, further, situations in fluid mechanics in which (1.1) is the appropriate model equation. Although the coefficient α of quadratic nonlinearity is normally positive, so that only compression shocks are formed, there are fluids (see the experiments of Borisov *et al.* (1983) on Freon-13) in which $\alpha < 0$ and expansion shocks are generated, whereas in other fluids α may vanish along some curve in a thermodynamic space. In that case, compression and expansion shocks are both permissible, and indeed both generated, and in the neighbourhood of the curve on which $\alpha = 0$ the modified Burgers equation (1.1) governs the motion, as shown by Cramer & Kluwick (1984). (In all of the above, the coefficient α is defined by the statement that in a suitable reference frame, with (x, t) space and time coordinates and u an appropriate wave variable, a right-running lossless simple wave satisfies $\partial u/\partial t + \alpha u\, \partial u/\partial x = 0$.)

With regard to terminology, we follow the custom now accepted in dispersive wave theory, whereby 'modified' refers to a change in nonlinearity from VV_τ to V^2V_τ, as in the Korteweg–de Vries and modified Korteweg–de Vries equations, reserving the term 'generalized' for changes, to Burgers equation, for example, arising from linear effects such as those associated with geometry or density stratification.

One may immediately ask what the position is with regard to exact solutions of (1.1). A travelling-wave solution $V = \phi(\tau - c_0 X)$ is of course obtainable, and will feature prominently in this paper. A similarity solution $V = X^{-\frac{1}{4}}\, g(\tau/X^{\frac{1}{2}})$ can also be found (Nariboli & Lin 1973) but has no relevance here. We have not yet been able to find any further exact solutions, despite attempts to use various 'direct' methods of the kinds proposed by Hirota (1976), Rosales (1978), Matsuno (1984) and others. We are also not encouraged in the search by a proof (Nimmo & Crighton 1982) that Burgers equation itself and the linear diffusion equation are the only parabolic equations possessing Bäcklund transformations onto themselves or onto any other parabolic equation, this proof allowing arbitrary (X, τ) dependence both in the differential equations and in the Bäcklund transformations. In the absence, then, both of a linearizing transformation and of an auto-Bäcklund transformation (that might at least lead to an exact N-shock solution), we seem forced to use either perturbation methods or numerical methods to understand the physical mechanisms contained in (1.1) and their balances for various magnitudes of (V, X, τ). In this paper we study (1.1) analytically with the aid of matched asymptotic expansions, backing up our asymptotic analysis with direct finite-difference calculations of the wave form and numerical integrations of the shock trajectories.

In previous papers (Crighton & Scott 1979; Nimmo & Crighton 1986), asymptotic solutions to various generalized Burgers equations were obtained in this way for small values of the diffusivity (ϵ in (1.1)) and $O(1)$ values of the other parameters involved, but allowing the independent variables to take all values of interest, including arbitrarily large values with respect to ϵ^{-1}. Thus we were able to analyse fairly fully the entire evolution of a wave that is fully nonlinear in the limit $\epsilon \to 0$ for finite values of (X, τ), and in particular, to follow the wave into its 'old age', in which it decays under linear mechanisms alone. Detailed calculations were presented for initial distributions $V(0, \tau)$ of N-wave and sinusoidal form; these distributions are of great interest in themselves (with applications to sonic booms and underwater parametric arrays, for example) and serve to illustrate most of the effects that seem to arise. In the present paper, the corresponding programme will be carried out for the modified Burgers equation (1.1), asymptotic solutions being based on the limit $\epsilon \to 0$ and detailed results being presented for N-wave and sinusoidal initial distributions.

The radically different feature of (1.1) (as compared with the ordinary Burgers equation) is that if one insists that data for the 'lossless' solutions just outside shock waves must be determined directly via characteristics from the initial data, then in general a singularity will develop at a finite time within the shock wave itself, and there will be no acceptable shock transition between the desired values of V. We stipulate here that a non-singular shock transition is mandatory, and examine then whether a consistent picture can be built up. This is shown to be possible if initial values are used to feed a value $V = H(X)$ via characteristics to one side of the shock, the value $V = G(X)$ on the other then being determined simply by the condition $H + 2G = 0$ for X greater than some finite range that is calculated for the cases considered. The signal value $V = G = -\frac{1}{2}H$ is then propagated in a 'lossless' fashion along characteristics; however, the gradient $\mathrm{d}\tau/\mathrm{d}X$ of the characteristic emanating from the initial data at $X = 0$ changes from H^2 to $\frac{1}{4}H^2$ through the shock, so that the shock sends out refracted characteristics with a factor $(\frac{1}{4})$ gradient change. These refracted characteristics are, in fact, tangential to the shock path, so that, once this new phase of the motion is reached, the shock is actually the envelope of the family of refracted characteristics. The refracted characteristics may then impinge upon another shock at which, in general, no acceptable transition can occur unless the condition $H + 2G = 0$ is again imposed. For the sinusoidal initial distribution one then has, for moderate finite times, a pattern of multiple refractions, by the periodic shocks, of characteristics starting from $X = 0$. This refraction process is investigated in detail for both initial data considered, and the leading-order solutions for the wave form are obtained in, essentially, all (X, τ) domains of interest (the exception being a region in which it appears that nothing short of the full equation (1.1) with all terms comparable is adequate).

Transition to old age is then examined, as the shocks thicken and lose their steady Taylor-like form (and also diffuse far from the location at which 'weak shock theory' (Whitham 1974, ch. 9) locates them for finite times). At ranges $O(\epsilon^{-2})$ and $O(\epsilon^{-1})$ for the N-wave and sinusoidal initial profiles, respectively, the lossless portions and shocks no longer have separate identities, and the whole wave is described by the full equation (1.1), for which no applicable solution is known. This prevents us from fully analysing the final decay into old age, but much of the ultimate structure can nevertheless be obtained.

To substantiate our view of the wave structure for finite times we present, in §4, some finite-difference calculations of the wave form that support the imposition of the condition

$H+2G = 0$ (although not as decisively as one might hope, because when this condition is imposed the shock matches algebraically with the lossless flow on one side, rather than exponentially, which means that it is not clear how to read off the values of H, G accurately from computed results). The discussion of the shock structure under the condition $H+2G = 0$ is then continued, in §5, in relation to Lax's (1973) 'generalized entropy condition'. Lax's work proves that certain inequalities must be satisfied by the shocks amplitudes of a 'lossless' problem

$$\left.\begin{aligned}\frac{\partial U}{\partial X}+\frac{\partial}{\partial \tau}f(U) &= 0,\\ U(0,\tau) &= U_0(\tau)\end{aligned}\right\}$$

if those are to correspond to the limiting behaviour, as $\epsilon \to 0$, of the (unique) solution to

$$\left.\begin{aligned}\frac{\partial U_\epsilon}{\partial X}+\frac{\partial}{\partial \tau}f(U_\epsilon) &= \epsilon\frac{\partial^2 U_\epsilon}{\partial \tau^2}\\ U_\epsilon(0,\tau) &= U_0(\tau)\end{aligned}\right\}.$$

We show in §5 that the Lax inequalities themselves do not require the condition $H+2G = 0$ to be identically satisfied, but that that condition is (a) necessary for an acceptable set of matched asymptotic solutions to our problem and (b) not merely compatible with the Lax inequalities but actually a limiting form of them.

The paper ends with a brief discussion, and with Appendixes giving the derivation of (1.1) in one representative case, and with an outline of the calculation of higher-order terms. Not surprisingly, this raises some difficulties we are unable to resolve, stemming from our inability to fully analyse the motion in one or two regions of (X, τ) space even to leading order, and the position is likely to remain unchanged until a general method of solving (1.1) is found.

2. The N-wave initial distribution

The form of modified Burgers equation that arises naturally in the type of boundary value problem considered in Appendix A is

$$\frac{\partial V}{\partial X}+V^2\frac{\partial V}{\partial \tau} = \epsilon\frac{\partial^2 V}{\partial \tau^2}, \tag{2.1}$$

in which V is a dimensionless signal, the time-like variable X is actually a range from some origin of excitation, and τ is a phase variable based on the small signal propagation speed. In this section we study the evolution with range of the N-wave initial profile depicted in figure 1.

2.1. *The lossless solution*

An outer expansion in (X, τ) variables starts with

$$V = V_0(X,\tau)+o(1) \tag{2.2}$$

as $\epsilon \to 0$, and the 'lossless solution' V_0 satisfies

$$\frac{\partial V_0}{\partial X}+V_0^2\frac{\partial V_0}{\partial \tau} = 0 \tag{2.3}$$

with $\qquad V_0(0,\tau) = q(\tau)$

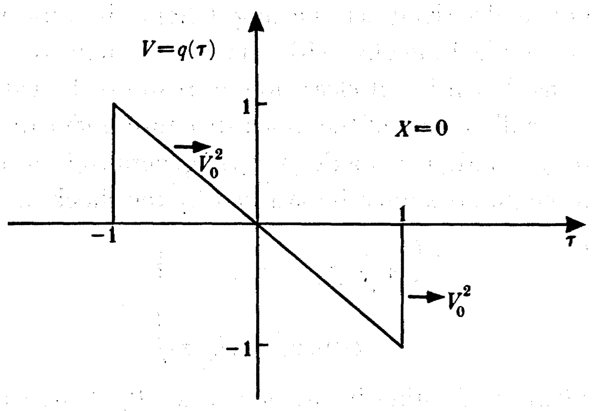

FIGURE 1. Initial N-wave profile.

and q the initial profile. The characteristics are straight lines, all with positive gradient, regardless of the form of the initial profile. In terms of a characteristic variable $p(X, \tau)$ defined by

$$\tau = p + Xq^2(p) \tag{2.4}$$

the solution is $V_0 = q(p)$, and may be given in explicit form as

$$\left.\begin{array}{lll} 0, & |\tau| > 1 & (|p| > 1) \\ {[(\tau+1)/X]^{\frac{1}{2}}}, & -1 < \tau < X-1 & (p = -1) \\ {[1-(1+4X\tau)^{\frac{1}{2}}]/2X}, & X-1 < \tau < X+1 & (-1 < p < +1) \\ -[(\tau-1)/X]^{\frac{1}{2}}, & 1 < \tau < X+1 & (p = 1). \end{array}\right\} \tag{2.5}$$

The solution V_0 defined in this way is depicted in figure 2.

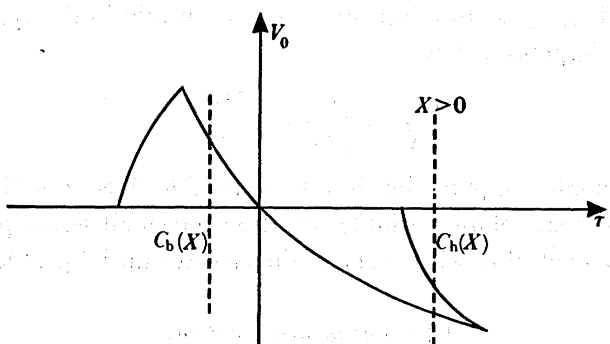

FIGURE 2. Lossless profile of N-wave for $X > 0$.

2.2. *Shock formation*

When the lossless solution becomes triple valued, a shock discontinuity must be fitted at an appropriate phase location. A triple-valued solution arises immediately (i.e. at $X = 0$) at $\tau = 1$, so that for $X \geqslant 0$ there is a 'head shock' moving into the undisturbed region, implying then that the arc $V_0 = -[(\tau-1)/X]^{\frac{1}{2}}$ for $1 < \tau < X+1$ is completely redundant. The phase location of the head shock is denoted by $\tau = C_h(X)$, as in figure 2. One might think that because of the nature of the initial profile there should be a shock towards the back of the wave, at $\tau = C_b(X)$ say, again cutting out an arc of the lossless wave. However, such a shock would not satisfy the generalized entropy condition to be discussed in §5 below; see Lax (1973) for a similar example.

A second or 'tail' shock does, however, form behind the head shock at a later stage, where cumulative nonlinear steepening on the arc for which $X-1 < \tau < X+1$ produces infinite gradient

$$\partial V_0/\partial\tau = -1/(1+4X\tau)^{\frac{1}{2}}$$

of the solution $V_0 = (2X)^{-1}[1-(1+4X\tau)^{\frac{1}{2}}]$. This first occurs at $X = \frac{1}{2}$, $\tau = -\frac{1}{2}$, where the value of V_0 is 1.

2.3. *Head-shock structure and dynamics*

Denote the value of V_0 just behind the head shock by $F(X)$; then from (2.5), shock amplitude and location are related by

$$C_h = XF^2 - F \tag{2.6}$$

with

$$C_h(0) = 1, \quad F(0) = -1.$$

We find another relation between C_h and F by analysing the shock structure with the aid of a shock variable

$$S_h^* = (\tau - C_h(X))/\epsilon$$

and a shock expansion of the form (because $F = O(1)$)

$$V(X, S_h^*, \epsilon) = V_0^*(X, S_h^*) + \epsilon V_1^*(X, S_h^*) + \dots. \tag{2.7}$$

The equation for V_0^* can be integrated once to

$$\partial V_0^*/\partial S_h^* = \tfrac{1}{3}(V_0^* - \alpha)(V_0^* - \beta)(V_0^* + \alpha + \beta), \tag{2.8}$$

where $\alpha(X)$, $\beta(X)$ are related to the arbitrary function of X arising from the integration, and to the shock propagation 'speed'

$$C_h'(X) = (\mathrm{d}\tau/\mathrm{d}X)_{\text{shock}}$$

by

$$C_h'(X) = \tfrac{1}{3}(\alpha^2 + \alpha\beta + \beta^2), \tag{2.9}$$

this of course being the expected result for shock propagation with signal levels α, β on either side of the shock in a medium with cubic nonlinearity (see Whitham 1974). The matching conditions to the outer wave are

$$V_0^* \to 0 \quad \text{as} \quad S_h^* \to +\infty,$$

$$V_0^* \to F(X) \quad \text{as} \quad S_h^* \to -\infty,$$

so that

$$\alpha(X) = F(X), \quad \beta(X) = 0, \tag{2.10}$$

and the integration can be completed to yield

$$V_0^* = F/\{1 + \exp[\tfrac{2}{3}F^2(S_h^* - S_{h_0}^*)]\}^{\frac{1}{2}}, \tag{2.11}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

where $S^*_{h_0}(X)$ is a 'correction due to diffusivity' to the shock location of weak-shock theory (see Lighthill 1956; Crighton & Scott 1979). This function is left undetermined by first-order matching of V_0^* to V_0. Equation (2.11) represents the analogue of the steady Taylor (1910) solution for the structure of a weak shock propagating into a medium at rest; in it, convection and diffusion are balanced, and the range variable X enters only parametrically, in $F(X)$ and $S^*_{h_0}(X)$.

Equations (2.9) and (2.10) give

$$C_h'(X) = \tfrac{1}{3}F^2(X), \tag{2.12}$$

and elimination of $C_h(X)$ between (2.6) and (2.12) gives a cubic

$$\tfrac{4}{3}XF^3 - F^2 + 1 = 0 \tag{2.13}$$

for the shock amplitude, of which the relevant solution is

$$\begin{aligned} F &= \frac{1}{4X}[1-2\cos\tfrac{1}{3}\{\arccos\,(24X^2-1)\}], \quad 0 \leqslant X \leqslant \tfrac{1}{2\sqrt{3}}, \\ &= \frac{1}{4X}[1-2\cosh\tfrac{1}{3}\{\operatorname{arcosh}\,(24X^2-1)\}], \quad X \geqslant \tfrac{1}{2\sqrt{3}}. \end{aligned} \tag{2.14}$$

The shock location $C_h(X)$ can now be found from (2.6).

These determinations of $F(X)$, $C_h(X)$, do not, in fact, retain their validity for all $O(1)$ values of X, and actually take quite different forms for $X > X_2 = 90$, as will shortly be seen. In addition to this, a complete analysis of the $O(1)$ head shock structure requires determination of the shock displacement, $S^*_{h_0}(X)$, due to diffusivity. This can only be achieved through higher-order matchings (or, possibly, through the judicious use of an integral conservation law, as in Crighton & Scott 1979), and these matters are all considered in outline in Appendix B, and in detail by Lee-Bapty (1981).

2.4. *Tail-shock structure and dynamics*

The tail-shock discontinuity is drawn between the lossless arcs given (from (2.4)) by

$$\tau_L = -1 + XV_0^2 \quad \text{on the left,}$$

and

$$\tau_R = -V_0 + XV_0^2 \quad \text{on the right,}$$

as in figure 3. Denoting the signal level to the right of the shock by $G(X)$, that on the left by $H(X)$, and the shock phase location by $\tau = C_t(X)$, these give

$$\left.\begin{aligned} C_t &= XH^2 - 1, \\ C_t &= XG^2 - G, \end{aligned}\right\} \tag{2.15}$$

with starting values $C_t(\tfrac{1}{2}) = -\tfrac{1}{2}$, $G(\tfrac{1}{2}) = H(\tfrac{1}{2}) = 1$. For the shock structure, an equation of the form (2.8) again describes the leading order solution, but with S_t^* replacing S_h^*, and where the matching conditions

$$V_0^* \to G(X) \quad \text{as} \quad S_t^* \to +\infty,$$

$$V_0^* \to H(X) \quad \text{as} \quad S_t^* \to -\infty,$$

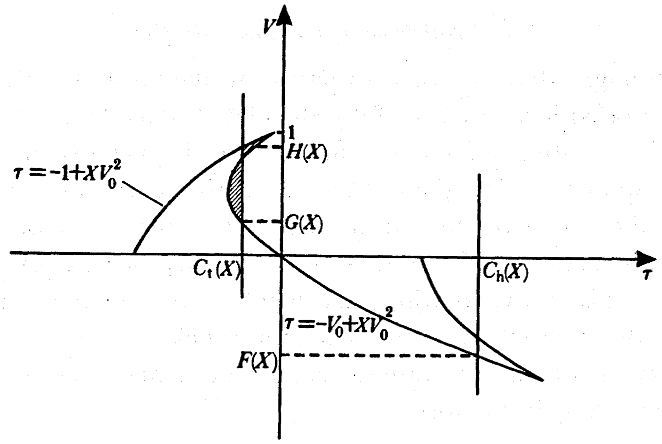

FIGURE 3. Head and tail shocks in lossless wave.

imply that $\alpha(X) = G(X)$, $\beta(X) = H(X)$, and hence that

$$C_t'(X) = \tfrac{1}{3}\{G^2(X)+G(X)\,H(X)+H^2(X)\}. \tag{2.16}$$

Equations (2.15) and (2.16) form a complete set for the determination (with the stated starting values) of C_t, G and H, and the shock structure is then determined by the implicit integral of (2.8),

$$(V_0^*-G)^{(G+2H)}\,(V_0^*-H)^{-(2G+H)}\,(V_0^*+G+H)^{-(H-G)}$$
$$= \exp\{-\tfrac{1}{3}(H-G)\,(G+2H)\,(2G+H)\,[S_t^*-S_{t_0}^*(X)]\} \tag{2.17}$$

in which $S_{t_0}^*(X)$ is another undetermined shock displacement due to diffusivity.

The integrations to determine G, H and C_t are facilitated by the application of Whitham's 'equal areas' rule (Whitham 1974, ch. 2) requiring the shock discontinuity to cut off lobes of equal area, as shown in figure 3, and expressed analytically in the form

$$(H-G)\,C_t(X) = \int_G^1 \tau_R\,\mathrm{d}V_0 - \int_H^1 \tau_L\,\mathrm{d}V_0$$

or

$$(H-G)\,C_t(X) = \tfrac{1}{2}-H+\tfrac{1}{2}G^2+\tfrac{1}{3}X(H^3-G^3) \tag{2.18}$$

after the integrations are performed. Equation (2.18) satisfies the initial conditions, and is equivalent (after differentiation and elimination of C_t with the aid of (2.15)) to (2.16). Use of (2.15) to eliminate C_t and H from (2.18) then gives a quartic in G,

$$(G-1)\left\{24G^3-\frac{39}{X}G^2+\left(\frac{18}{X}+\frac{16}{X^2}\right)G+\left(\frac{9}{X}-\frac{16}{X^2}\right)\right\}=0$$

of which the root $G = 1$ (and hence $H = 1$) is the trivial solution representing the vertex of the N-wave over the range $0 < X < \frac{1}{2}$ before the tail shock has actually formed. Thus for the genuine tail shock ($X > \frac{1}{2}$) the cubic applies, and has the one real root

$$G = \frac{1}{X}\left[\tfrac{13}{24}-(X-\tfrac{41}{144})^{\frac{1}{2}}\sinh\{\tfrac{1}{3}\operatorname{arsinh}[\tfrac{3}{2}X^2-\tfrac{25}{24}X+\tfrac{299}{1728}]/(X-\tfrac{41}{144})^{\frac{3}{2}}\}\right], \tag{2.19}$$

from which the values of H, C_t follow from (2.15).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

2.5. *Breakdown of tail-shock structure*

It might now be thought that the wave structure (in the form of lossless regions separated by thin steady-state shocks) has now been fully elucidated to leading order for $X = O(1)$, and that what remains is the study of the transition into old age (as in the programme, for generalized Burgers equations with quadratic nonlinearity, carried out by Crighton & Scott (1979)). That is not the case here, nor can it be expected to be the case for any nonlinearity $(V^n)_\tau$ with $n \geqslant 3$, and in fact when X exceeds a finite value X_1 the wave exhibits one of the most interesting features of this study, in which the whole nature of the outer field is radically changed, along with the internal structure of the tail shock.

The way, and range, at which this dramatic change takes place can be found by examining the tail-shock equation (2.8) in the form

$$\partial V_0^*/\partial S_t^* = \tfrac{1}{3}[V_0^* - G(X)]\,[V_0^* - H(X)]\,[V_0^* - I(X)],$$

where $I(X) = -G(X) - H(X)$, and considering the sketches of figure 4. In figure 4*b* it can be seen that, when the roots are such that $H > G > I$ (unbracketed symbols), there is a smooth

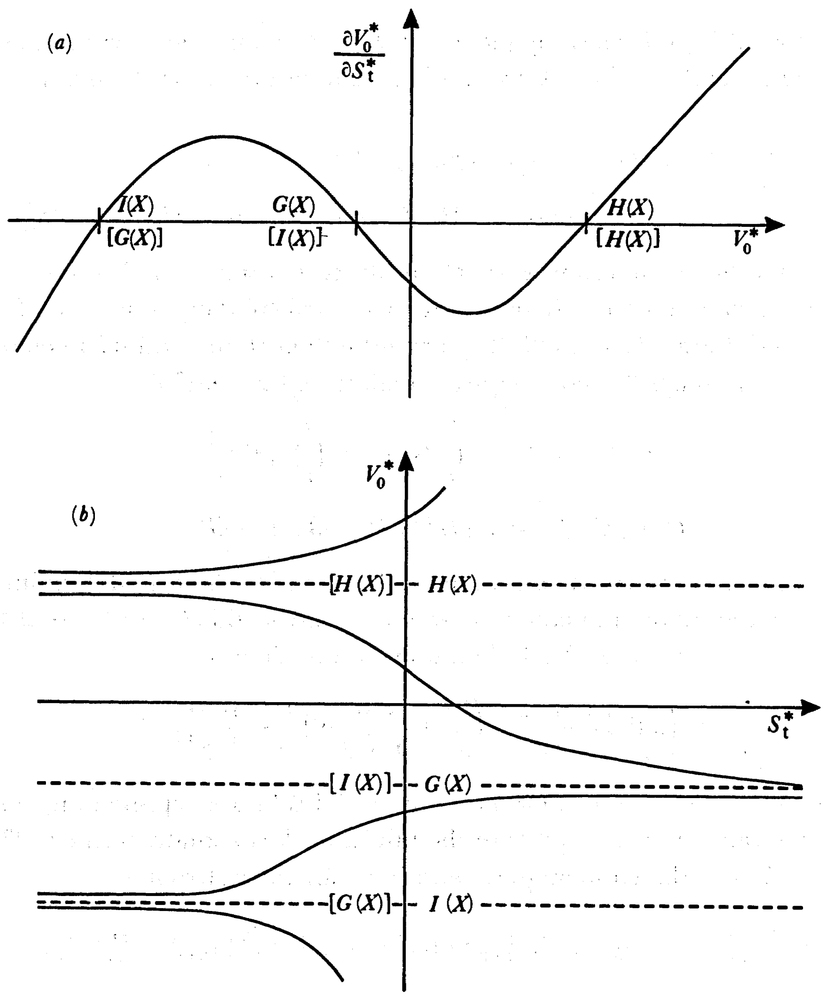

FIGURE 4. (*a*) Sketch of the function $(\partial V_0^*/\partial S_t^*)(V_0^*)$. (*b*) Sketch of the function $V_0^*(S_t^*)$.

transition from H to G, with negative $\partial V_0^*/\partial S_t^*$. However, if the positions of G and I were reversed, corresponding to the bracketed symbols, so that $H > I > G$, there would be no non-singular shock transition from H to G. Thus if $I = -G-H > G$, or $H+2G < 0$, there would cease to be an acceptable shock transition.

The condition for an acceptable (i.e. non-singular) shock structure is

$$H+2G \geqslant 0 \tag{2.20}$$

and is equivalent to the restriction imposed by Lax's generalized entropy condition (see §5). To show that there does in fact exist at least one X such that $H+2G = 0$, we observe that $G = H = 1$ when $X = \frac{1}{2}$ and that an investigation of the asymptotics of the equations for small $(X-\frac{1}{2})$ and for large X shows that

$$\begin{aligned} G &= 1-3(X-\tfrac{1}{2})+O(X-\tfrac{1}{2})^2, \\ H &= 1+O(X-\tfrac{1}{2})^2 \qquad \text{as } X-\tfrac{1}{2}\downarrow 0, \\ H &\sim -G \sim \tfrac{1}{2}(3/X)^{\frac{1}{3}} \qquad \text{as } X\to\infty, \end{aligned}$$

which proves the point. Let X_1 be the first such value of X. The significance of what occurs at range X_1 can be appreciated more fully from inspection of the characteristics plot for the lossless solution, given in figure 5. Write (2.16) in the form

$$C_t'(X)-G^2(X) = \tfrac{1}{3}(H-G)\,(H+2G), \tag{2.21}$$

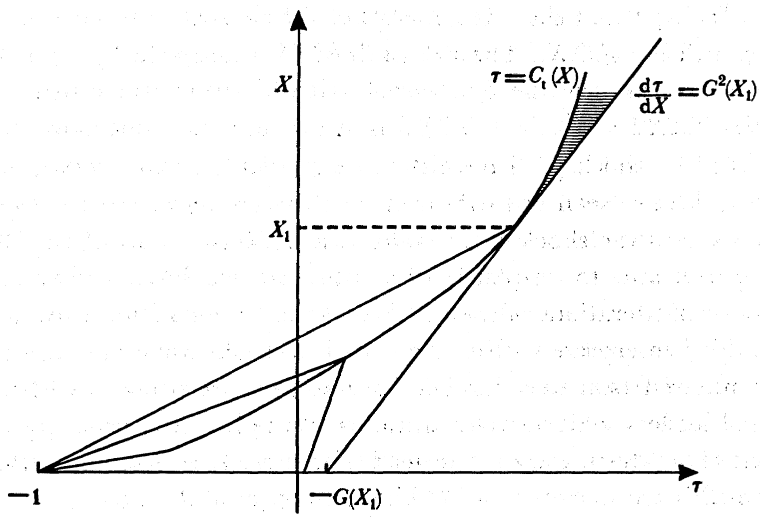

FIGURE 5. Characteristics and shock path in (τ, X) plane.

where, on the left, $C_t'(X)$ is the shock 'speed', the gradient of the shock path $\tau = C_t(X)$, and $G^2(X)$ is the gradient of the characteristic emanating from the initial data at phase $\tau = -G(X)$. Then, because $H+2G$ vanishes at X_1 we see that $C_t'(X_1) = G^2(X_1)$ and the shock path and characteristic touch at X_1. Beyond $X = X_1$ the shock path must remain to the left of this tangent characteristic (for otherwise characteristics emanating from $\tau = -1$ initially would be able to cross the tangent characteristic) and it appears then that for $X > X_1$ there is a region (shaded in figure 5) of lossless dynamics that cannot be reached by characteristics coming directly from the initial line $X = 0$. To avoid this loss of information, and to preserve an acceptable shock structure for $X > X_1$, the following argument must be used to determine the lossless solution in the shaded region.

Observe first that the shock path must not only lie to the left of the tangent characteristic at X_1, but must satisfy

$$C_t'(X) - G^2(X) \leqslant 0$$

for all $X \geqslant X_1$ (otherwise the lossless solution to the left of the shock would be multivalued). Second, the requirement that there be a non-singular shock transition between the lossless signals H, G (with our convention that $H > G$) implies, from (2.20) and (2.21), that

$$C_t'(X) - G^2(X) \geqslant 0,$$

and the two requirements can be simultaneously satisfied if, and only if,

$$C_t'(X) = G^2(X) = \tfrac{1}{4}H^2(X) \tag{2.22}$$

or

$$H + 2G = 0$$

for all $X \geqslant X_1$. This means that for $X > X_1$ all information to the shock from the right is lost, so that the second of equations (2.15), $C_t = XG^2 - G$, no longer applies. All subsequent information to the right of the shock originates from the initial data at $\tau = -1$ and is passed through the shock in such a way that the signal level on the right, G, is $(-\frac{1}{2})$ that, H, on the left. This signal level $(-\frac{1}{2}H)$ is then propagated in 'lossless' fashion along the characteristics with $d\tau/dX = (\frac{1}{4}H^2)$, so that there is a *refraction* of the characteristics from $\tau = -1$ when they meet the shock path beyond X_1. The value of $d\tau/dX$ is multiplied by $(\frac{1}{4})$ as the characteristic passes through the shock, and the signal level carried by the characteristic acquires the factor $(-\frac{1}{2})$. Condition (2.22), $C_t'(X) = G^2(X)$ is, moreover, the condition that the refracted characteristic and the shock path have the same gradient, and therefore the new shock, for $X \geqslant X_1$, is the envelope (seen from the right) of the family of refracted characteristics. The refraction process, and the shock as an envelope, are illustrated in figure 6.

It is also appropriate to emphasize that although we have arrived at the condition $H + 2G \geqslant 0$ from considerations of the shock structure, the condition can also be simply (and perhaps preferably) interpreted with respect to the lossless wave operator. Difficulties must always be encountered (sometimes implying instability, sometimes, as here, involving indeterminacy of the lossless solution) unless the shock speed, in the usual physical variables, is greater than the characteristic velocity ahead of the shock, and less than that behind. For the present problem this requirement (see Whitham 1974, ch. 3) is that

$$H^2 \geqslant C_t'(X) \geqslant G^2,$$

and $C_t' = G^2$ is achieved at X_1, whereas $C_t' < G^2$ would follow if the value of G for $X > X_1$ were supplied by the initial data. Therefore, considerations of the lossless solution alone, with a shock discontinuity, imply that $C_t' \geqslant G^2$, whereas $C_t' \leqslant G^2$ also remains in force, and so condition (2.22) follows without consideration of the detailed shock structure.

The tail-shock relations are much simplified for $X > X_1$; we have (together with (2.22)) just

$$C_t(X) = XH^2 - 1,$$

and

$$C_t'(X) = \tfrac{1}{4}H^2, \tag{2.23}$$

with solutions
$$\left.\begin{aligned} C_t(X) &= -1+d^2X^{\frac{1}{4}},\\ H(X) &= dX^{-\frac{3}{8}}. \end{aligned}\right\} \tag{2.24}$$

To determine the integration constant d, return to (2.15) and (2.18) (which hold for $X \leqslant X_1$), eliminating X and $C_t(X)$ to give

$$4H^2+(G-3)\,H+(G-3)\,G = 0,$$

and then setting $H = -2G$ to arrive at

$$G(G+\tfrac{1}{5}) = 0.$$

The root $G = 0$ refers to $X = \infty$, so that

$$G(X_1) = -\tfrac{1}{5}, \quad H(X_1) = \tfrac{2}{5}, \tag{2.25}$$

from which it readily follows that

$$C_t(X_1) = \tfrac{3}{5}, \quad X_1 = 10, \tag{2.26}$$

and hence
$$d = \tfrac{2}{5}(10)^{\frac{3}{8}} \approx 0.94855.$$

Eventually, this change in the tail-shock dynamics affects the head shock, because the tangential characteristic with $\mathrm{d}\tau/\mathrm{d}X = G^2(X_1)$ eventually intersects the head shock path $\tau = C_h(X)$, at $X = X_2$ say, beyond which range information to the left of the head shock can no longer come from the initial data to supply the shock amplitude $F(X)$. Instead, this information comes along the refracted characteristics sent out by the tail shock; see figure 6. Thus, for $X > X_2$ the cubic equation (2.13) can no longer be used to determine $F(X)$, and what is needed is the relation between points on the head shock and those on the tail-shock path to which they are linked by the refracted characteristics. Equivalently, if the characteristic which leaves the tail shock at $(C_t(\xi), \xi)$ intersects the head shock at $(C_h(\eta), \eta)$, then what is needed is the function $\eta(\xi)$; see figure 6 for a diagram.

To determine this function we have the relation

$$F(\eta) = -\tfrac{1}{2}H(\xi)$$

following from (2.22), the tail-shock condition

$$C_t'(\xi) = \tfrac{1}{4}H^2(\xi),$$

the head-shock condition
$$C_h'(\eta) = \tfrac{1}{3}F^2(\eta)$$

and the relation
$$C_h(\eta)-C_t(\xi) = \tfrac{1}{4}H^2(\xi)\,(\eta-\xi)$$

following from the refraction of the characteristic. The solution for $\eta(\xi)$ subject to $\eta(X_1) = X_2$ is

$$\eta = \xi+(X_2-X_1)\,\frac{H^3(X_1)}{H^3(\xi)}-\frac{1}{H^3(\xi)}\int_{X_1}^{\xi} H^3(X)\,\mathrm{d}X,$$

which simplifies greatly on use of (2.24) to

$$\eta = 9\xi+(X_2-9X_1)\,(\xi/X_1)^{\frac{9}{8}}.$$

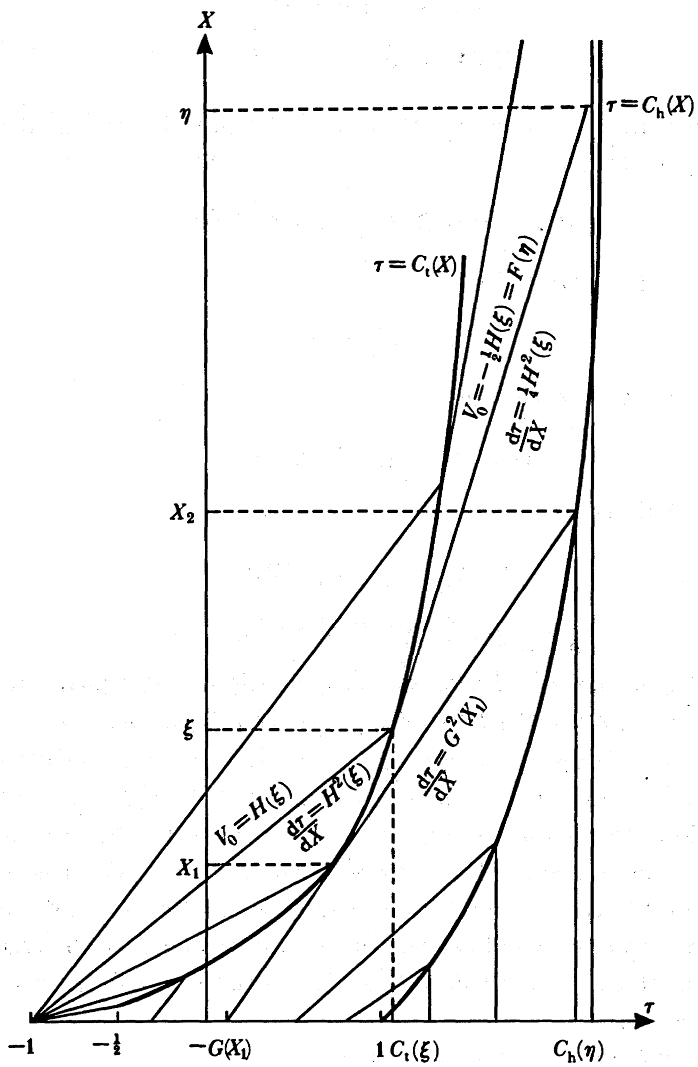

FIGURE 6. Characteristics plot for N-wave.

Now $F(X_2) = -\frac{1}{2}H(X_1) = -\frac{1}{5}$, and because F must presumably be continuous across $X = X_2$ it follows that $(X_2, F(X_2) = -\frac{1}{5})$ must satisfy the cubic (2.13), from which we have

$$X_2 = 90 = 9X_1. \tag{2.27}$$

This leads to the very simple expressions

$$\eta = 9\xi,$$

$$F(\eta) = -\tfrac{1}{2}H(\xi) = -\tfrac{1}{2}H(\tfrac{1}{9}\eta),$$

or

$$F(X) = -e_1/X^{\frac{3}{8}}, \tag{2.28}$$

where $e_1 = \frac{1}{2}(9)^{\frac{3}{8}}\, d \approx 1.0811$, and, from (2.12),

$$C_h(X) = 3^{\frac{1}{2}} d^2 X^{\frac{1}{4}} - 1. \tag{2.29}$$

These provide the strength and location of the head shock for $X > X_2$, whereas (2.24) provides the corresponding quantities for the tail shock for $X > X_1$. The leading order lossless solution between the shocks, for

$$\tau < G^2(X_1)\,X - G(X_1) \equiv \tfrac{1}{25}X + \tfrac{1}{5}$$

is given by

$$V_0 = -\tfrac{1}{2}H(\xi),$$

$$\tau - C_t(\xi) = \tfrac{1}{4}H^2(\xi)\,(X-\xi)$$

(see figure 6). Substitution for $H(\xi)$ and $C_t(\xi)$ gives

$$V_0 = -\tfrac{1}{2}d\xi^{-\frac{3}{8}},$$

where

$$\xi - \frac{4}{3}\frac{(\tau+1)}{d^2}\,\xi^{\frac{3}{4}} + \tfrac{1}{3}X = 0,$$

and this leads to the explicit representation

$$V_0 = -\tfrac{1}{2}d\left\{\left(\frac{\tau+1}{3d^2}\right)\left[1+(1+Z)^{\frac{1}{2}} - \left(2\left(1+\frac{1}{\sqrt{(1+Z)}}\right)-Z\right)^{\frac{1}{2}}\right]\right\}^{-\frac{3}{2}}$$

in which

$$Z = \frac{3d^4X^{\frac{1}{3}}}{(\tau+1)^2}\cosh\left\{\frac{1}{3}\operatorname{arcosh}\left[\frac{(\tau+1)^2}{d^4X^{\frac{1}{3}}}\right]\right\}. \tag{2.30}$$

Before going on to analyse the transition into old age, we should note two consequences of the change in shock dynamics that takes place at $X = X_1$.

First, the shock amplitude $G(X)$ on the right has discontinuous gradient at X_1, for when $X < X_1$, $(2XG-1)\,\mathrm{d}G/\mathrm{d}X = C_t'(X) - G^2(X)$ and so

$$\lim_{X\uparrow X_1} \mathrm{d}G/\mathrm{d}X = 0,$$

whereas for $X > X_1$, $\mathrm{d}G/\mathrm{d}X = -\tfrac{1}{2}\mathrm{d}H/\mathrm{d}X$ and

$$\lim_{X\downarrow X_1} \mathrm{d}G/\mathrm{d}X = \tfrac{3}{400}.$$

Second, the internal shock structure is significantly different for $X > X_1$. Then, the cumbersome expression (2.17) can be simplified by setting $G(X) = -\tfrac{1}{2}H(X)$ to give

$$V_0^* = \tfrac{1}{4}H(X)\,(1-3\tanh y_t), \tag{2.31}$$

where

$$1+\mathrm{e}^{2y_t}+2y_t = \tfrac{3}{4}H^2(X)\,(S_t^* - S_{t_0}^*(X)). \tag{2.32}$$

This implicit relation for y_t gives rise to an unusual shock structure, corresponding to the fact that the cubic in (2.8) now has $H(X)$ as a simple root but $-\tfrac{1}{2}H(X)$ as a double root.

For large negative S_t^* the linear term in y_t dominates, so that $2y_t \sim \tfrac{3}{4}H^2S_t^*$ and hence

$$V_0^* \sim H - \tfrac{3}{2}H\exp\{\tfrac{3}{4}H^2S_t^*\}, \tag{2.33}$$

so that the shock tends exponentially to the outer wave at the upper end, as would be expected. However for large positive S_t^* the exponential term dominates, so that here

$$V_0^* \sim -\tfrac{1}{2}H + \frac{2}{HS_t^*}, \tag{2.34}$$

i.e. at the lower end the shock tends only *algebraically* to the outer wave. This feature, not normally associated with Taylor shocks, is significant, because for $X < X_1$ the approach to the outer wave is exponential at both ends, and to accommodate this change in behaviour at the lower end of the shock a new local region is required about $V = -\frac{1}{2}H(X)$.

The discontinuity in $\mathrm{d}G/\mathrm{d}X$ and the behaviour of the shock solution, which are consequences of the changed leading order outer solution, generate higher-order problems outlined in Appendix B. These effects are not particular to the N-wave problem, but arise in any boundary-value problem from which the new type of shock (i.e. one with signal levels in the ratio $2:-1$) evolves, such as the sinusoidal initial condition problem considered in §3.

The leading-order outer problem has now been solved for $X = O(1)$, and we turn now to the large X non-uniformities.

2.6. *Transition to old age*

The asymptotic separation of the wave form into lossless regions separated by thin shocks is permissible provided three conditions are met. These conditions are (Crighton & Scott 1979) (i) that the shock thickness must be small compared with the scale of the lossless wave portions; (ii) that the 'correction due to diffusivity' to the shock location, calculated according to 'weak-shock theory' (Whitham 1974, ch. 9) or its equivalent for higher-order nonlinearity (as in the present problem), should be small; and (iii) that the steady Taylor-type shock solution V_0^* should itself remain valid as a leading-order approximation.

We investigate whether these assumptions continue to be met indefinitely (i.e. for large X) by the solutions obtained for $O(1)$ values of X. Consider first the head shock. As regards (i), the ratio of shock thickness to overall scale is

$$[V_0^*/(\partial V_0^*/\partial \tau)]/C_h(X) = O(\epsilon X^{\frac{1}{2}})$$

with estimates based on (2.11), (2.28) and (2.29). For (ii), the correction $S_{h_0}^*(X)$ to the head-shock location remains undetermined, for reasons explained in Appendix B. To test condition (iii) we calculate a second term V_1^* in the shock expansion $V = V_0^* + \epsilon V_1^* + o(\epsilon)$; details are given in the thesis by Lee-Bapty (1981), and we find

$$\epsilon V_1^*/V_0^* \sim \epsilon F'(X)/F^5(X)$$
$$= O(\epsilon X^{\frac{1}{2}})$$

with the implication that V_0^* is not a correct leading order approximation when $X = O(\epsilon^{-2})$ or larger.

For the tail shock, precisely the same estimates can be established in regard to conditions (i) and (iii), while the correction $S_{t_0}^*(X)$ required for (ii) again remains undetermined.

It is conceivable that a non-uniformity could arise through violation of condition (ii) for one or both of the shocks before conditions (i) and (iii) are violated. We know of no instance, however, in which condition (ii) alone is first violated (see Crighton & Scott 1979; Lee-Bapty 1981; Nimmo & Crighton 1986), and we therefore argue that the first non-uniformity in the $X = O(1)$ picture arises when $X = O(\epsilon^{-2})$ and involves the whole wave (whereas if (i) were not violated the non-uniformity associated with (iii) would be confined in the first place to the shocks alone, as happens in the case of spherical N-waves with quadratic nonlinearity, discussed by Crighton & Scott (1979)). This suggests scalings for a new region defined by

$$X = \epsilon^{-2}\bar{X}, \quad V = \epsilon^{\frac{3}{4}}\bar{V},$$

the second of these being dictated by the shock amplitudes $H(X)$, $F(X)$, and

$$\tau = \epsilon^{-\frac{1}{2}}\bar{\tau}$$

dictated by the value of $\tau = C_{\mathrm{h}}(X)$. Then with $\bar{V} = \bar{V}_0 + o(1)$ we find that in $(\bar{X}, \bar{\tau})$ variables the motion at leading order is governed by the full modified Burgers equation

$$\frac{\partial \bar{V}_0}{\partial \bar{X}} + \bar{V}_0^2 \frac{\partial \bar{V}_0}{\partial \bar{\tau}} = \frac{\partial^2 \bar{V}_0}{\partial \bar{\tau}^2}. \tag{2.35}$$

Matching conditions are the following; to the lossless wave,

$$\lim_{\epsilon \to 0} [\epsilon^{\frac{3}{4}} \bar{V}_0(\bar{X} = \epsilon^2 X, \bar{\tau} = \epsilon^{\frac{1}{2}}\tau)] = \begin{cases} 0 & \text{for} \quad \bar{\tau} < 0, \\ (\tau/X)^{\frac{1}{2}} & \text{for} \quad 0 < \bar{\tau} < d^2 \bar{X}^{\frac{1}{4}} \\ -\tfrac{1}{2}d\left\{\left(\dfrac{\tau}{3d^2}\right)\left[1 + (1+\bar{Z})^{\frac{1}{2}} - \left(2\left(1 + \dfrac{1}{\sqrt{(1+\bar{Z})}}\right) - \bar{Z}\right)^{\frac{1}{2}}\right]\right\}^{-\frac{3}{2}} & \\ \qquad \text{for} \quad d^2\bar{X}^{\frac{1}{4}} < \bar{\tau} < 3^{\frac{1}{2}} d^2 \bar{X}^{\frac{1}{4}}, & \\ 0 & \text{for} \quad \bar{\tau} > 3^{\frac{1}{2}} d^2 \bar{X}^{\frac{1}{4}}, \end{cases}$$

where
$$\bar{Z} = \frac{3d^4 X^{\frac{1}{2}}}{\tau^2} \cosh\left\{\frac{1}{3}\,\mathrm{arcosh}\left(\frac{\tau^2}{d^4 X^{\frac{1}{2}}}\right)\right\};$$

to the head shock,

$$\lim_{\epsilon \to 0} [\epsilon^{\frac{3}{4}} \bar{V}_0(\bar{X} = \epsilon^2 X, \bar{\tau} = \epsilon^{\frac{1}{2}} C_{\mathrm{h}}(X) + \epsilon^{\frac{3}{4}} S_{\mathrm{h}}^*)] = (1 + \exp \bar{y}_{\mathrm{h}})^{-\frac{1}{2}} F(X) \quad \text{for} \quad \bar{\tau} \sim 3^{\frac{1}{2}} d^2 \bar{X}^{\frac{1}{4}},$$

where
$$\bar{y}_{\mathrm{h}} = \tfrac{2}{3} e_1^2 \{X^{-\frac{3}{4}} S_{\mathrm{h}}^* - \lim_{X \to \infty} [X^{-\frac{3}{4}} S_{\mathrm{h}_0}^*(X)]\};$$

and to the tail shock

$$\lim_{\epsilon \to 0} [\epsilon^{\frac{3}{4}} \bar{V}_0(\bar{X} = \epsilon^2 X, \bar{\tau} = \epsilon^{\frac{1}{2}} C_{\mathrm{t}}(X) + \epsilon^{\frac{3}{4}} S_{\mathrm{t}}^*)] = \tfrac{1}{4} H(X)\,(1 - 3\tanh \bar{y}_{\mathrm{t}}) \quad \text{for} \quad \bar{\tau} \sim d^2 \bar{X}^{\frac{1}{4}},$$

where $\bar{y}_{\mathrm{t}}$ satisfies

$$1 + \exp(2\bar{y}_{\mathrm{t}}) + 2\bar{y}_{\mathrm{t}} = \tfrac{3}{4} d^2 \{X^{-\frac{3}{4}} S_{\mathrm{t}}^* - \lim_{X \to \infty} [X^{-\frac{3}{4}} S_{\mathrm{t}_0}^*(X)]\}.$$

In the absence of any general method of solving (2.35), the function $\bar{V}_0$ cannot be found, and a numerical attack would be formidably difficult in view of the complicated matching conditions. However, we can argue that, as in the corresponding problem with quadratic non-linearity, the dissipative mechanism will, for $X \gg \epsilon^{-2}$, reduce the amplitude of the whole wave, so that then the heat conduction equation

$$\partial \bar{V}_0 / \partial \bar{X} = \partial^2 \bar{V}_0 / \partial \bar{\tau}^2$$

will apply, which has a dipole solution (required because $\int_{-\infty}^{+\infty} V \,\mathrm{d}\tau = \int_{-\infty}^{+\infty} q(\tau)\,\mathrm{d}\tau = 0$)

$$\bar{V}_0 = -2\bar{c}\,\frac{\partial}{\partial \bar{\tau}}\left\{\bar{X}^{-\frac{1}{2}} \exp\left(-\frac{(\bar{\tau} - \bar{\tau}_0)^2}{4\bar{X}}\right)\right\}$$

$$= \bar{c}\left(\frac{\bar{\tau} - \bar{\tau}_0}{\bar{X}^{\frac{3}{2}}}\right) \exp\left\{-\frac{(\bar{\tau} - \bar{\tau}_0)^2}{4\bar{X}}\right\}, \tag{2.36}$$

in which $\bar{c}$, $\bar{\tau}_0$ are purely numerical constants. The ultimate decay of the wave is in this way determined essentially up to a purely numerical multiplicative factor in the amplitude.

Apart from a sketch of the higher-order terms given in Appendix B, this completes our study of the N-wave problem, and we move next to the case of a sinusoidal initial signal.

3. The sinusoidal initial distribution

The problem to be studied here is governed by the modified Burgers equation (2.1) with the initial (boundary) value

$$V(0,\tau) = \sin\tau.$$

We use the condition

$$V(X,\tau+\pi) = -V(X,\tau), \tag{3.1}$$

initiated in the boundary condition and preserved by the differential equation, to confine attention to one half-cycle when convenient.

3.1. *The lossless solution*

As in §2 we assume $V = V_0 + o(1)$, so that

$$\frac{\partial V_0}{\partial X} + V_0^2 \frac{\partial V_0}{\partial \tau} = 0, \quad V_0(0,\tau) = \sin\tau,$$

with solution

$$\left.\begin{aligned} V_0 &= \sin p, \quad \tau = p + X\sin^2 p, \\ \text{or} \qquad V_0 &= \sin\{\tau - XV_0^2\}. \end{aligned}\right\} \tag{3.2}$$

The characteristics are, of course, straight and all of positive gradient, producing the nonlinear steepening illustrated in figure 7. Multivalued solutions are produced by (3.2) beyond the range, $X = 1$, where the derivatives $\partial V_0/\partial\tau$, $\partial V_0/\partial X$ first become infinite. The characteristic variable which produces the infinity at $X = 1$ is $p = -\frac{1}{4}\pi$ or $\frac{3}{4}\pi$, corresponding to phase $\tau = -\frac{1}{4}\pi + \frac{1}{2}$ or $\frac{3}{4}\pi + \frac{1}{2}$, within the fundamental period. The values of V_0 are $-\frac{1}{\sqrt{2}}$, $+\frac{1}{\sqrt{2}}$, respectively.

To obtain a single-valued wave form beyond $X = 1$, shocks must be fitted, and only the one initially in $[0, \pi]$ need be considered in view of condition (3.1). Let $\tau = C(X)$ define the shock

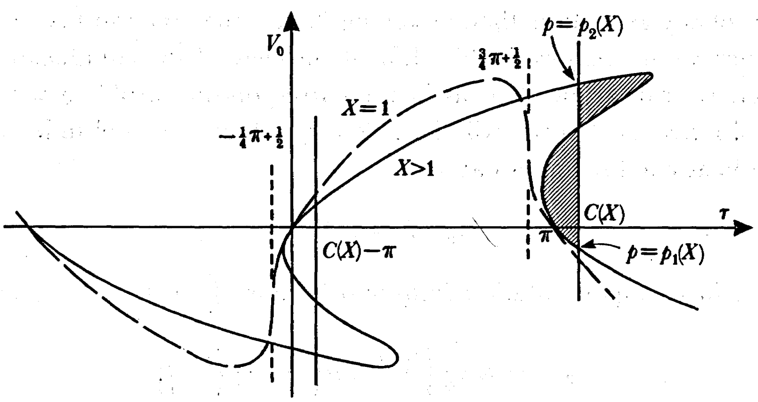

Figure 7. Lossless profiles for sinusoidal wave: ----, $X = 1$; ——, $X > 1$. Shocks at $\tau = C(X)$ and $\tau = C(X) - \pi$ are shown.

location, and let $p_1(X)$, $p_2(X)$ denote the characteristic variables ahead of, and behind the shock, respectively; see figure 7. Then the characteristic relation (3.2) gives

$$\left.\begin{aligned} C(X) &= p_1 + X\sin^2 p_1, \\ C(X) &= p_2 + X\sin^2 p_2, \end{aligned}\right\} \tag{3.3}$$

with initial conditions

$$C = \tfrac{3}{4}\pi + \tfrac{1}{2}, \quad p_1 = p_2 = \tfrac{3}{4}\pi \quad \text{at} \quad X = 1. \tag{3.4}$$

Implementation of the equal areas rule produces a third equation,

$$C(X)\,(\sin p_2 - \sin p_1) = \int_{\sin p_1}^{\sin p_2} \tau\,\mathrm{d}V_0$$

$$= (p_2 \sin p_2 + \cos p_2) - (p_1 \sin p_1 + \cos p_1) + \tfrac{1}{3}X(\sin^3 p_2 - \sin^3 p_1), \tag{3.5}$$

which can be cast in the form

$$2(\sin^2 p_1 + \sin p_1 \sin p_2 + \sin^2 p_2)\,(p_1 - p_2) + 3(\cos p_1 - \cos p_2)\,(\sin p_1 + \sin p_2) = 0 \tag{3.6}$$

with the aid of (3.3). Observe that direct differentiation of (3.5) produces the standard relation (Whitham 1974)

$$C'(X) = \tfrac{1}{3}\{\sin^2 p_1 + \sin p_1 \sin p_2 + \sin^2 p_2\}$$

for the shock propagation 'speed' in terms of the signal levels on either side of the shock.

Equations (3.3) and (3.6) form a complete set for determination of $C(X)$, $p_1(X)$, $p_2(X)$, although because of their strong nonlinearity we have only been able to solve these equations numerically. The numerical solution of (3.3) and (3.6) is of no particular interest in itself until we come to discuss the possible development of an internal shock singularity, as in §2.5, so that no details of the numerical solution will be given now.

3.2 *The shock structure*

Take a reference frame moving with the shock, so that τ is replaced by $S = \tau - C(X)$, and (2.1) becomes

$$\frac{\partial V}{\partial X} + \{V^2 - C'(X)\}\frac{\partial V}{\partial S} = \epsilon\frac{\partial^2 V}{\partial S^2}.$$

Then we examine the shock structure in terms of X and $S^* = S/\epsilon$, with

$$V(X, S^*, \epsilon) = V_0^*(X, S^*) + o(1),$$

and it is found that the solution for V_0^* is as for the tail shock in the N-wave problem. Thus (2.17) applies again, except that S^* is written for S_t^* and $S_0^*(X)$ (an undetermined shock displacement due to diffusivity) for $S_{t_0}^*(X)$. Matching the solution (2.17) to the lossless outer solution V_0 gives

$$G(X) = \sin p_1, \quad H(X) = \sin p_2. \tag{3.7}$$

3.3. *The embryo-shock region*

Before determining the range for which the above shock solution is valid we look at the 'embryo shock region' (Crighton & Scott 1979), the transition region in which the shock first forms before developing its steady Taylor-like structure. The region is a local one, centred on

$$X = 1, \quad \tau = C(1) = \tfrac{3}{4}\pi + \tfrac{1}{2}, \quad V = \tfrac{1}{\sqrt{2}}.$$

It is again appropriate to scale the moving-frame variables (X, S), and (by analysing the circumstances in which a second term, ϵV_1, of the outer solution becomes comparable with V_0) the required scalings are found to be

$$\left.\begin{aligned} V &= \tfrac{1}{\sqrt{2}}(1+\epsilon^{\frac{1}{4}}\hat{V}) \\ S = \epsilon^{\frac{3}{4}}\hat{S}, &\quad X = 1+\epsilon^{\frac{1}{2}}\hat{X}. \end{aligned}\right\} \tag{3.8}$$

Then an expansion $\hat{V} = \hat{V}_0 + o(1)$ gives

$$\frac{\partial \hat{V}_0}{\partial \hat{X}} + \hat{V}_0 \frac{\partial \hat{V}_0}{\partial \hat{S}} = \frac{\partial^2 \hat{V}_0}{\partial \hat{S}^2}, \tag{3.9}$$

so that the ordinary Burgers equation with quadratic nonlinearity governs the process of shock formation for the cubic nonlinearity of the modified Burgers equation.

The matching conditions are, to the outer lossless wave,

$$\lim_{\epsilon \to 0} \tfrac{1}{\sqrt{2}}\left\{1+\epsilon^{\frac{1}{4}}\hat{V}_0\left(\hat{X} = \frac{(X-1)}{\epsilon^{\frac{1}{2}}}, \hat{S} = \frac{S}{\epsilon^{\frac{3}{4}}}\right)\right\} = \tfrac{1}{\sqrt{2}}(1-p') \tag{3.10}$$

where $p' \sim p - \frac{3}{4}\pi$ is given by the cubic

$$\tfrac{2}{3}p'^3 - (X-1)\,p' - S = 0, \tag{3.11}$$

and to the shock

$$\lim_{\epsilon \to 0} \tfrac{1}{\sqrt{2}}\left\{1+\epsilon^{\frac{1}{4}}\hat{V}_0\left(\hat{X} = \frac{(X-1)}{\epsilon^{\frac{1}{2}}}, \hat{S} = \epsilon^{\frac{1}{4}}S^*\right)\right\} = \tfrac{1}{\sqrt{2}} - \tfrac{1}{2}\sqrt{3}\,(X-1)^{\frac{1}{2}} \tanh\{(\tfrac{3}{8})^{\frac{1}{2}}(X-1)^{\frac{1}{2}}S^*\} \tag{3.12}$$

for $\hat{X} > 0$. Condition (3.12) comes from expansion of the shock relation (2.17) for $X-1 \to 0$, and shows that initially the shock takes on the usual antisymmetric 'tanh' profile associated with shocks of the Burgers equation type.

The required solution to (3.9) can be found by use of the Cole–Hopf transformation, and the way in which the matching conditions are applied has been given by Crighton & Scott (1979). The result, almost identical to that for the embryo shock region for a sinusoidal initial disturbance evolving according to the ordinary Burgers equation, is

$$V = \tfrac{1}{\sqrt{2}}\left\{1-2\epsilon^{\frac{1}{4}}\frac{\partial}{\partial \hat{S}}\ln\int_{-\infty}^{\infty}\exp\left(-\tfrac{1}{12}q^4+\tfrac{1}{2}q\hat{S}+\tfrac{1}{4}q^2\hat{X}\right)\mathrm{d}q\right\}+o(\epsilon^{\frac{1}{4}}). \tag{3.13}$$

3.4. *Change in wave structure*

As in §2, the present combination of lossless arcs governed by (3.2) with matched Taylor-type shocks whose dynamics and structure were analysed in §3.2 will remain a valid representation of the wave for $X = O(1)$ unless the condition

$$H+2G \geqslant 0 \tag{3.14}$$

on the outer flow is violated. Now a study of the large X asymptotics of (3.3) and (3.6) reveals that

$$H(X) \sim -G(X) \sim a/X^{\frac{1}{3}} \quad (a>0)$$

so that condition (3.14) is indeed violated for some finite X, $X = X_1$, say. By solving (3.3) and (3.6) numerically we find

$$\left.\begin{aligned} X_1 = 9.6013590, \quad & H(X_1) = 0.62206017, \\ G(X_1) = -0.31103008, \quad & C'(X_1) = 0.096739713, \\ C(X_1) = 4.3867020 = 1.39633063\pi. & \end{aligned}\right\} \tag{3.15}$$

At X_1 there is again tangency of the shock path and the characteristic carrying the value $G(X_1)$ from the initial data, and lossless information to the right of the shock appears to be lost for $X > X_1$. This is resolved, and a non-singular shock transition between $H(X)$ and $G(X)$ assured, if the arguments of §2 are followed, imposing the condition

$$H + 2G = 0 \tag{3.16}$$

for all $X \geqslant X_1$ to determine $G(X)$ given $H(X)$. Again, the characteristics to the right of the shock are tangent to the shock for $X \geqslant X_1$.

For $X > X_1$, the shock relations then reduce to

$$\left.\begin{aligned} H(X) &= \sin p_2, \\ C(X) = p_2 + X\sin^2 p_2, \quad & C'(X) = \tfrac{1}{4}\sin^2 p_2, \end{aligned}\right\} \tag{3.17}$$

but this set does not, however, determine the shock dynamics for *all* $X > X_1$. Eventually, because of the periodicity of the motion, information feeding the left side of the shock ('left' here referring either to the wave form illustrated in figure 7 or the characteristic diagram of figure 8) no longer comes directly via characteristics from the initial line (as it always can do for the N-wave tail-shock of §2) but emanates from the right side of the shock in the previous half cycle. Again, for larger X, the information for the shock in the previous half-cycle may itself also not come immediately from the initial line, but along a characteristic that is refracted repeatedly on passing through a whole series of shocks, each refraction multiplying the signal carried by the characteristic by the factor $(-\frac{1}{2})$. This multiple refraction process is depicted in figure 8.

To derive analytic expressions for the outer flow for all $X = O(1)$, first define a set of X-values $\{X_n\}$, $n \geqslant 1$, such that the characteristic leaving the right side of a shock path at range X_n intersects the shock path for the next half-cycle (i.e. the one with τ increased by π) at range X_{n+1}. Similarly define the set $\{\xi_n\}$, $X_{n-1} < \xi_{n-1} < X_n$, by stating that the characteristic emanating from the shock path at $(C(\xi_{n-1}), \xi_{n-1})$ in the (τ, X) plane of figures 8 and 9 intersects the shock path of the next half cycle at $(\pi + C(\xi_n), \xi_n)$. In view of the antisymmetry condition (3.1), we have the relation

$$H(\xi_n) = \tfrac{1}{2}H(\xi_{n-1}), \tag{3.18}$$

where $H(\xi_n)$ is the signal magnitude on the left of the shock at $X = \xi_n$. Then, because $H(\xi_1)$ is known for $X_1 < \xi_1 < X_2$ from the set (3.17), $H(X)$ will be known for all X once ξ_n can be related to ξ_{n-1}, and so eventually to ξ_1.

To achieve this, consider the region, shown in figure 9, between the two shocks given by $\tau = C(X)$ and $\tau = \pi + C(X)$. The leading-order lossless solution at a general point here can be

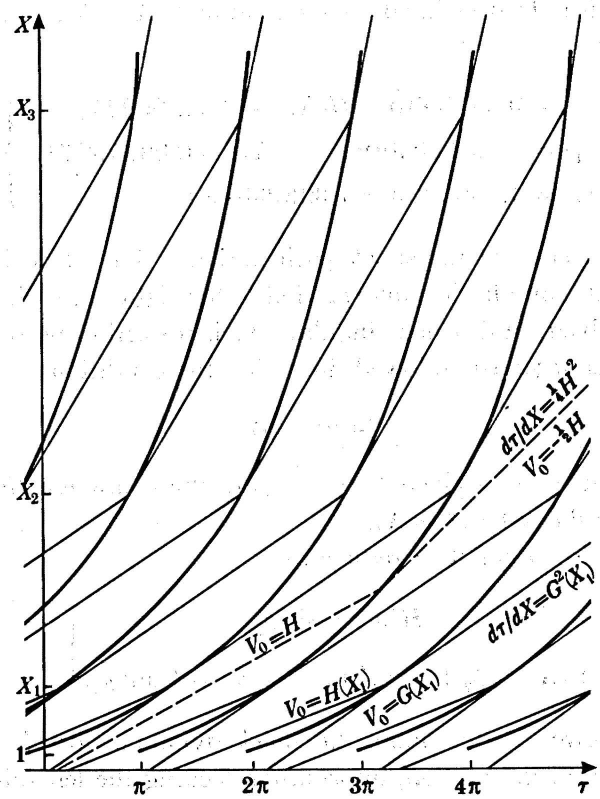

FIGURE 8. Characteristics plot for sinusoidal wave, showing multiple refraction of characteristics through the periodic pattern of shocks.

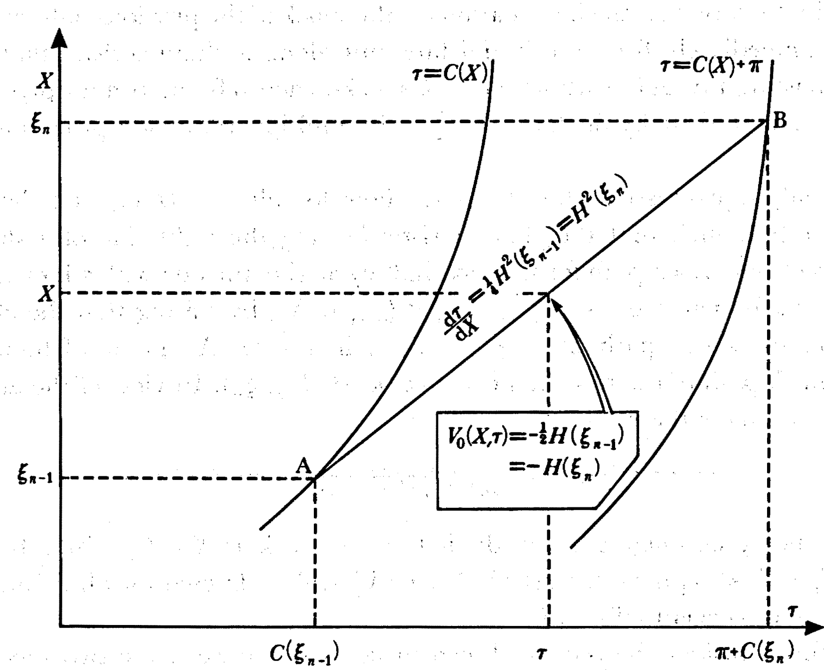

FIGURE 9. Region between consecutive shocks in sinusoidal wave.

given either in terms of the shock amplitude on the left, or of that on the right. Working from the left gives

$$\left.\begin{aligned} V_0(X,\tau) &= -\tfrac{1}{2}H(\xi_{n-1}), \\ \tau - C(\xi_{n-1}) &= \tfrac{1}{4}H^2(\xi_{n-1})\,(X-\xi_{n-1}), \\ C'(\xi_{n-1}) &= \tfrac{1}{4}H^2(\xi_{n-1}), \end{aligned}\right\} \tag{3.19}$$

whereas working from the right gives

$$\left.\begin{aligned} V_0(X,\tau) &= -H(\xi_n), \\ \pi + C(\xi_n) - \tau &= H^2(\xi_n)\,(\xi_n - X), \\ C'(\xi_n) &= \tfrac{1}{4}H^2(\xi_n). \end{aligned}\right\} \tag{3.20}$$

Now add the second equations of (3.19) and (3.20) to give

$$\pi + C(\xi_n) - C(\xi_{n-1}) = \tfrac{1}{4}H^2(\xi_{n-1})\,(\xi_n - \xi_{n-1}), \tag{3.21}$$

then differentiate with respect to ξ_{n-1} and substitute for $C'(\xi_{n-1})$ and $C(\xi_n)$ from (3.19) and (3.20) to arrive at

$$\frac{d\xi_n}{d\xi_{n-1}} + \frac{8}{3}\left[\frac{d}{d\xi_{n-1}}\ln H(\xi_{n-1})\right](\xi_n - \xi_{n-1}) = 0. \tag{3.22}$$

Alternatively, integrating $C'(X) = \frac{1}{4}H^2(X)$ from ξ_{n-1} to ξ_n gives

$$C(\xi_n) - C(\xi_{n-1}) = \frac{1}{4}\int_{\xi_{n-1}}^{\xi_n} H^2(X)\,dX,$$

which, subtracted from (3.21), gives the result

$$(\xi_n - \xi_{n-1})\,H^2(\xi_{n-1}) - \int_{\xi_{n-1}}^{\xi_n} H^2(X)\,dX = 4\pi. \tag{3.23}$$

Either of (3.22) or (3.23) determines ξ_n in terms of ξ_{n-1}, and hence ultimately in terms of ξ_1. And because $H(\xi_1)$ is known (numerically), the function $H(X)$ can be built up by using (3.18). This has been done numerically, with results displayed in figure 10.

Although we cannot determine $H(X)$ explicitly, its asymptotics for large X can be found. Substitution of the Ansatz $H \sim aX^{-\beta}$ in (3.18), (3.22) and (3.23) reveals that

$$\left.\begin{aligned} H(X) &\sim 2\,[\pi/(3-2\ln 2)]^{\frac{1}{2}}\,X^{-\frac{1}{2}}, \\ C(X) &\sim [\pi/(3-2\ln 2)]\ln X + C_\infty. \end{aligned}\right\} \tag{3.24}$$

Observe that the form for $H(X)$ is substantiated by the numerical results; C_∞ is a numerical constant that is not predicted by the asymptotic analysis but found from computation of $C(X)$, $H(X)$ to have the value -1.7291.

With these asymptotic forms for $H(X)$ and $C(X)$, ξ_{n-1} can now be eliminated from (3.19) to give the implicit form

$$V_0 \sim U(X,S), \quad -\pi < S < +\pi,$$

where $S = \tau - C(X)$, with

$$\left.\begin{aligned} U(X,S) &= (b^2U_1(S)/X)^{\frac{1}{2}}, \quad -\pi < S < 0, \\ &= -(b^2U_2(S)/X)^{\frac{1}{2}}, \quad 0 < S < \pi \end{aligned}\right\} \tag{3.25}$$

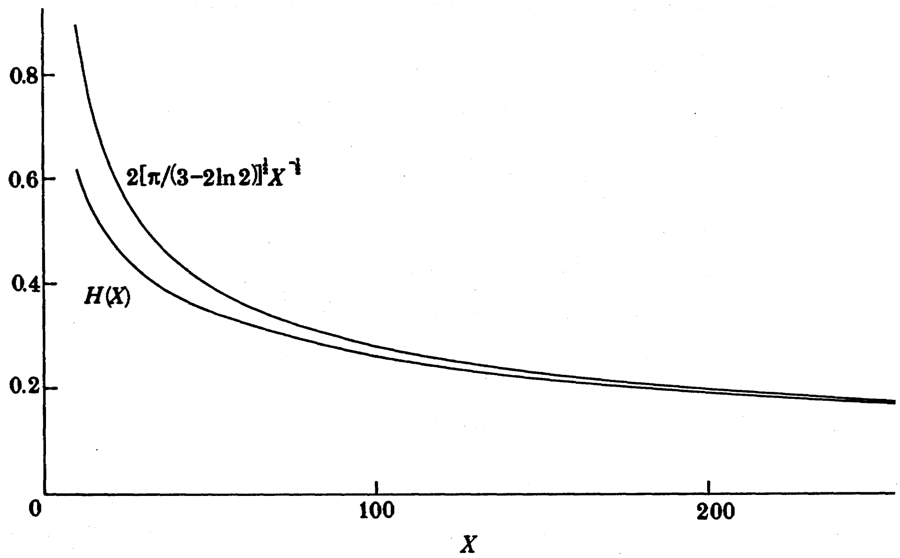

FIGURE 10. Numerically determined values of $H(X)$ and comparison with asymptotic prediction of equation (3.24).

and where $U_1(S)$ and $U_2(S)$ are defined by

$$\begin{aligned} b^2(U_1 - \ln \tfrac{1}{4}(U_1) - 4) &= S \quad \text{for} \quad 1 < U_1 < 4, \\ b^2(U_2 - \ln U_2 - 1) &= S \quad \text{for} \quad 1 < U_2 < 4. \end{aligned} \tag{3.26}$$

The constant b^2 appears in (3.24); $b^2 = \pi/(3-2\ln 2)$.

This completes discussion of the outer flow for all $X = O(1)$. As in §2, the shock structure has a neater, and interesting, form for $X > X_1$; replacing $G(X)$ by $-\frac{1}{2}H(X)$ gives, in (2.17),

$$V_0^*(X, S^*) = \tfrac{1}{4}H(X)\,(1 - 3\tanh y),$$

where

$$1 + e^{2y} + 2y = \tfrac{3}{4}H^2(X)\,(S^* - S_0^*(X)) \tag{3.27}$$

and $\tau = C(X) + \epsilon S^*$. As in §2, $V_0^* \to H$ exponentially in S^* as $S^* \to -\infty$, $V_0^* \to (-\frac{1}{2})H$ algebraically in S^* as $S^* \to +\infty$.

3.5. *Transition to old age*

To examine the transition into old age we consider the circumstances under which the criteria (i)–(iii) of §2.6 are violated. In the above section we have given full details of the leading-order outer solution and of the leading-order shock solution V_0^*; Lee-Bapty (1981) provides an expression for the term ϵV_1^* in the shock expansion, and Appendix B gives an estimate of the shock displacement due to diffusivity. By using these various expressions it is easy to show that all three criteria (i)–(iii) for the validity of the current representation in terms of lossless arcs and thin steady state shocks, are violated at the *same* range

$$X = O(\epsilon^{-1}).$$

Appropriate rescalings are then

$$X = \bar{X}/\epsilon, \quad V = \epsilon^{\frac{1}{2}}\,\bar{V}, \tag{3.28}$$

the latter following from the behaviour of $H(X)$ as $X \to \infty$. The phase variable τ remains unscaled, because the non-uniformity at $X = O(\epsilon^{-1})$ involves the whole of the wave. With the scalings (3.28), the leading-order wave function $\bar{V}_0$ satisfies the full modified Burgers equation

$$\frac{\partial \bar{V}_0}{\partial \bar{X}} + \bar{V}_0^2 \frac{\partial \bar{V}_0}{\partial \tau} = \frac{\partial^2 \bar{V}_0}{\partial \tau^2}, \tag{3.29}$$

with matching to the lossless wave expressed by

$$\lim_{\epsilon \to 0} [\epsilon^{\frac{1}{2}} \bar{V}_0\{\bar{X} = \epsilon X, \tau = C(X) + S\}] = U(X, S) \quad \text{for} \quad C(X) - \pi < \tau < \pi + C(X),$$

where $U(X,S)$ is defined in (3.25, 3.26), and matching to the shock (3.27) expressed by

$$\lim_{\epsilon \to 0} [\epsilon^{\frac{1}{2}} \bar{V}_0\{\bar{X} = \epsilon X, \tau = C(X) + \epsilon S^*\}] = \tfrac{1}{2}[\pi/(3 - 2\ln 2)]^{\frac{1}{2}} X^{-\frac{1}{2}} (1 - 3\tanh \bar{y}) \quad \text{for} \quad \tau \sim C(X),$$

where $\bar{y}$ satisfies

$$1 + e^{2\bar{y}} + 2\bar{y} = [3\pi/(3 - 2\ln 2)]\{(S^*/X) - (S_{00} \ln X + S_{01})\}$$

and S_{00}, S_{01} are identified in Appendix B. In the absence of any relevant solutions to (3.29) one can only observe that eventually, for $X \gg \epsilon^{-1}$, dissipation will reduce the wave amplitude to such an extent that the linear version of (3.29) will apply. Then a solution satisfying (3.1) is

$$\begin{aligned} V(X,\tau) &\sim \sum_{n=0}^{\infty} A_n e^{-(2n+1)^2 \epsilon X} \sin\{(2n+1)\tau - \alpha_n\} \\ &\sim A_0 e^{-\epsilon X} \sin(\tau - \alpha_0), \end{aligned} \tag{3.30}$$

where A_0, α_0 are undetermined purely numerical constants. If $\bar{V}_0$ could be determined, A_0 and α_0 could be found by letting $\bar{X} = \epsilon X \to \infty$ in $\bar{V}_0$. Observe (from (A 4) and (A 5)) that $\bar{X}$ is independent of the initial amplitude E_0 of the signal, as is τ, and therefore that the solution at range $O(\epsilon^{-1})$ is, from (3.28),

$$V = \frac{E}{E_0} = \left(\frac{\beta\omega}{3\alpha E_0^2}\right)^{\frac{1}{2}} \bar{V}_0(\bar{X}, \tau). \tag{3.31}$$

Here we have used the particular model (nonlinear electromagnetic waves) of Appendix A, but the result is general. It says that the signal E at range $X = O(\epsilon^{-1})$ is independent of the initial amplitude E_0, and so, that the signal has suffered the phenomenon of amplitude saturation, which is well known for the quadratic nonlinearity of the ordinary Burgers equation (for a detailed discussion of how this arises also in generalized Burgers equations with quadratic nonlinearity see Nimmo & Crighton (1986)). For a sinusoidal signal and the quadratic Burgers equation, the result that guarantees amplitude saturation is decay of V as X^{-1} at range ϵ^{-1}, ϵ being then inversely proportional to the initial amplitude. Here, with cubic nonlinearity, ϵ varies as E_0^{-2}, but the signal at range ϵ^{-1} decays only as $X^{-\frac{1}{2}}$ (see (3.25)) and there is therefore again the interesting and significant phenomenon of amplitude saturation. Observe also that the saturation does not apply merely to the old-age linear decay, where (3.30) and (3.31) predict, for the physical model of Appendix A,

$$E \sim \left(\frac{\beta\omega}{3\alpha}\right)^{\frac{1}{2}} A_0 \exp[-\tfrac{1}{2}\mu_0 c\beta\omega^2 x] \sin\left[\omega\left(t - \frac{x}{c} - \frac{\alpha_0}{\omega}\right)\right], \tag{3.32}$$

but also to the fully nonlinear phase of the motion in which (3.31) is the solution, with $\bar{V}_0$ a solution of the full MBE (3.29).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

4. Numerical results for the sinusoidal initial distribution

Here an attempt to substantiate the results of the preceding section is made by employing numerical techniques. We use a finite difference scheme to follow the evolution of $V(X, \tau)$ with X according to (2.1), starting from $V(0, \tau) = \sin \tau$. A three-level scheme is used, in which the derivatives at the centre of a 3×3 grid are approximated by using central differences that involve all nine mesh points. The derivatives $\partial V/\partial \tau$ and $\partial^2 V/\partial \tau^2$ are averaged over all three X-rows, but in the nonlinear term $V^2 \partial V/\partial \tau$ the value of V is taken to be that at the central point only.

This scheme can be thought of as an extension to the Crank–Nicholson (in which all values are averaged over two levels and which is unconditionally stable). The introduction of a third (middle) level and the retention of nonlinearity in that level alone ensure that the resulting simultaneous equations in the unknowns of the upper level are linear. Solution of this set of equations is quite simple, involving direct methods to invert a tri-diagonal matrix.

Initially an attempt was made to use the Crank–Nicholson scheme, but the indirect methods used to solve the resulting nonlinear equations took a long time to converge once a certain range X was exceeded; see Mitchell (1969, §§2.19–2.22) for an appraisal of both three-level and two-level schemes for nonlinear equations.

Of course, the disadvantage of three-level schemes is that unconditional stability is forfeited. However, the stability criterion for the three-level scheme can be interpreted as $\Delta \tau < \pi \epsilon$, where $\Delta \tau$ is the step size in τ, and this just means that the step length must be smaller than the shock thickness, i.e. that there must be at least one mesh point within the shock. We have taken $\Delta \tau = \frac{1}{2}\pi \epsilon$, which gives stability, whereas taking $\Delta \tau = \pi \epsilon$ was found to lead to instability. The dependence of stability on ΔX is more obscure; to be on the safe side we have taken ΔX as small as $\Delta \tau$, although hindsight suggests that this may have been unnecessarily small despite the fact that it makes the scheme consistent (formally) in its accuracy. To start the three-level scheme we need to know the solution on two levels, and for this we applied a Taylor series using $\partial V(0, \tau)/\partial X$ evaluated from (2.1).

The major influence on the running time of the program was the size of ϵ, and hence of ΔX, $\Delta \tau$. Because the aim of the computation was to give credence to the asymptotic theory for $\epsilon \to 0$, a compromise had to be reached in which ϵ was small enough for the asymptotic behaviour 'to bite', but not so small that the computing time became unrealistic. By comparing profiles of V against τ at fixed X for successively decreasing values of ϵ we were able to ascertain the stage at which the wave was tending to a limiting (asymptotic) form. We found that this stage was reached when $\epsilon = \frac{1}{400}$, and that this was not too small a number with which to compute.

The results of the computation are shown in figures 11 and 12. Figure 11 shows the series of profiles $V(X, \tau)$ for $X = 0(1)\,16$, and figure 12 the series for $X = 0(10)\,370$. As principal aim of the numerical work we wish to substantiate the claim that the shock amplitudes fall into the $2:-1$ ratio after a finite distance X_1. The amplitude $H(X)$ is well defined in each profile, but $G(X)$ is, unfortunately, less so, except for $X = 2, 3$. Two factors contribute to this; first, and most simple, as $V \to H(X)$ the profile turns through an angle greater than $\frac{1}{2}\pi$, whereas when $V \to G(X)$ the profile turns through an angle less than $\frac{1}{2}\pi$ (at least for $X \lesssim 30$); second, the theory does predict that, for $X > X_1$, the matching to $G(X)$ should only be algebraic, rather than exponential (cf. §3.4). To gain greater definition in the region about $G(X)$ we would need to take a much smaller value of ϵ.

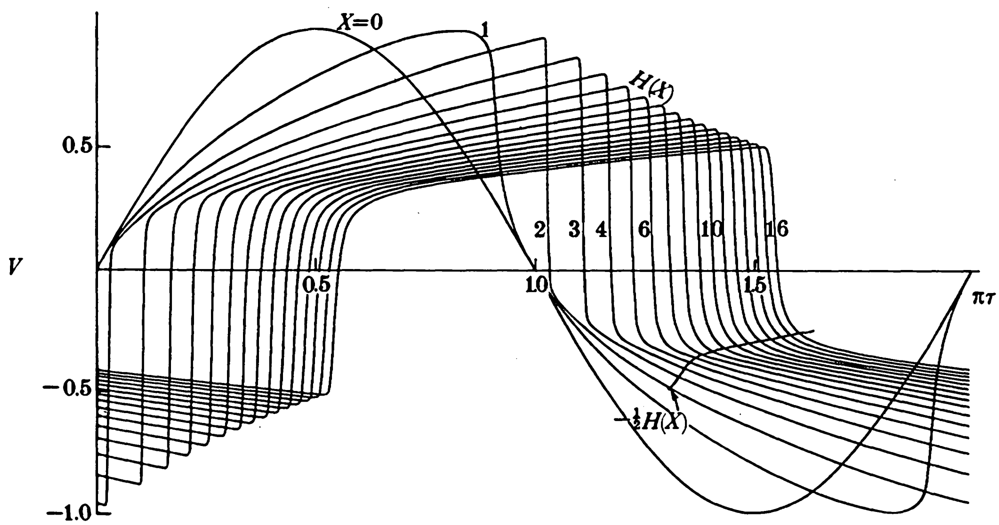

FIGURE 11. Profiles of V as function of τ for $X = 0(1)\ 16$; numerical determination as described in §4.

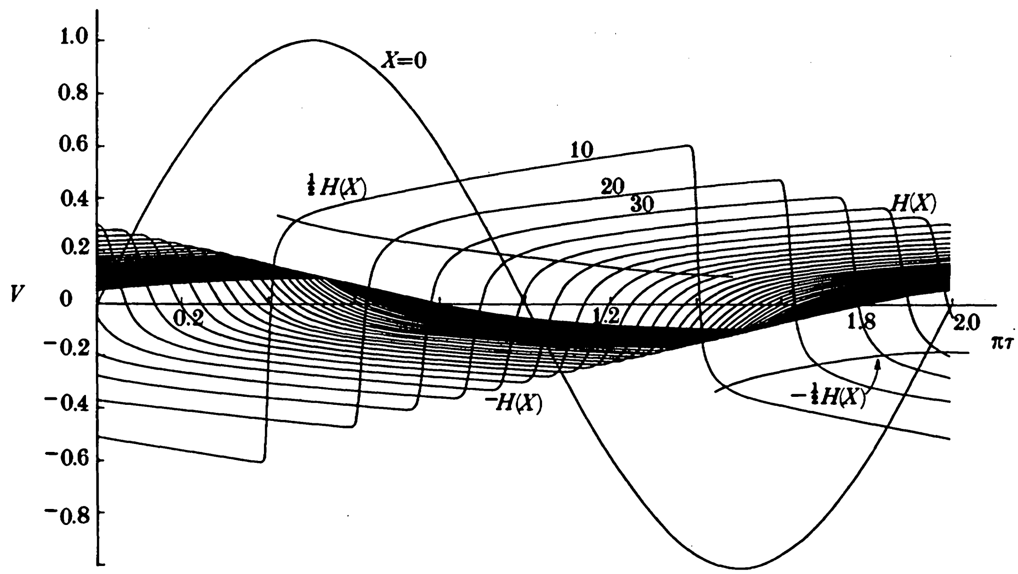

FIGURE 12. Profiles, as in figure 11, for $X = 0(10)\ 370$.

Although the profiles do not explicitly show the behaviour predicted by the theory, it can be seen, by considering the line drawn in figures 11 and 12 through the points with $V = -\frac{1}{2}H(X)$ (actually $+\frac{1}{2}H(X)$ in figure 12, but note the symmetry condition (3.1)) that qualitatively the theory does indeed give a good representation of the shock amplitudes for $X > X_1 \approx 9.6$. The same qualitative features can be seen in the wave forms given in Gorschkov *et al.* (1974, figure 2). In particular, the $2:-1$ ratio of shock amplitudes can be surmised to hold, although one cannot say more as the figures (as published there) contain no scales and no indication of the range X, and are evidently intended only to provide a qualitative idea of the type of wave form to be expected.

As X becomes large we see, in figure 12, how the shocks thicken and decrease in strength,

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

eventually smoothing out into the whole wave as predicted in §3.5. Already at $X = 370$ we see the wave tending to the sinusoidal form of the old-age régime, in agreement with the prediction of §3.5 that the thin shock waves should have disappeared by $X = O(\epsilon^{-1})$.

5. Relation to Lax's generalized entropy condition

Lax (1973) states that the solution to the lossless problem defined by

$$\left.\begin{aligned}\frac{\partial U}{\partial X}+\frac{\partial}{\partial \tau}f(U) &= 0,\\ U(0,\tau) &= U_0(\tau),\end{aligned}\right\} \tag{5.1}$$

can be uniquely defined as the limit as $\epsilon \to 0$ of the solution, U_ϵ, of the dissipative problem

$$\left.\begin{aligned}\frac{\partial U_\epsilon}{\partial X}+\frac{\partial}{\partial \tau}f(U_\epsilon) &= \epsilon\frac{\partial^2 U_\epsilon}{\partial \tau^2},\\ U_\epsilon(0,\tau) &= U_0(\tau),\end{aligned}\right\} \tag{5.2}$$

and he proves that such a 'distribution' solution to (5.1) must satisfy a generalized entropy condition. This condition states that, for a discontinuous solution of (5.1), with $U = G(X)$ to the right of the discontinuity and $U = H(X)$ on the left, (i) if $G < H$, the curve $f(U)$ on $[G, H]$ must lie below the chord drawn between $(G, f(G))$ and $(H, f(H))$ as shown in figure 13*a*; whereas (ii) if $G > H$, the curve $f(U)$ on $[H, G]$ must lie above the chord drawn between $(H, f(H))$ and $(G, f(G))$, as shown in figure 13*b*. Analytically, these imply

$$f(U)-\left(\frac{f(H)-f(G)}{H-G}\right)U+\left(\frac{Gf(H)-Hf(G)}{H-G}\right)\begin{cases}\leqslant 0 & \text{for} \quad G\leqslant U\leqslant H\\ \geqslant 0 & \text{for} \quad H\leqslant U\leqslant G,\end{cases} \tag{5.3}$$

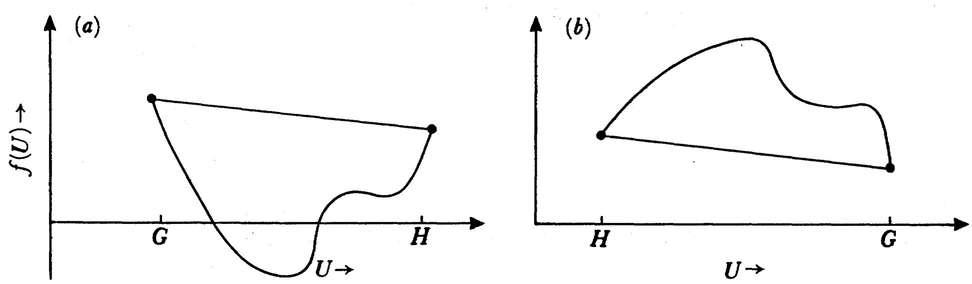

Figure 13 (*a*) The function $f(U)$ for $H > G$. (*b*) The function $f(U)$ for $H < G$.

and we observe (Whitham 1974, p. 31) that the gradient of the chord is the shock 'speed' $C'(X)$. Then the use of Taylor series about the end points G and H yields, from (5.3), the results

$$f'(G) \leqslant C'(X) \leqslant f'(H), \quad (H > G),$$

$$f'(H) \leqslant C'(X) \leqslant f'(G), \quad (H < G),$$

which confirm that the shock speed $C'(X)$ must lie between the characteristic velocities, $f'(G)$, $f'(H)$, of points on the wave at either side of the discontinuity (cf. the discussion of §2 above).

Example (*a*). $f(U) = \frac{1}{2}U^2$, for which (5.2) is the ordinary Burgers equation (Lighthill 1956) for plane gasdynamic waves. Condition (5.3) for discontinuities is

$$\begin{aligned}(U-G)(U-H) &\leqslant 0 \quad \text{for} \quad G \leqslant U \leqslant H \\ &\geqslant 0 \quad \text{for} \quad H \leqslant U \leqslant G,\end{aligned}$$

of which the first is automatically satisfied, whereas the second can never be. This implies that there can be no discontinuities for which $H < G$, i.e. no rarefaction shocks, only shocks of compression. Thus the application of (5.3) is equivalent to the familiar assertion (e.g. Lighthill 1956) that, on physical grounds, entropy must increase across a gasdynamic shock.

Example (*b*). $f(U) = \frac{1}{3}U^3$, for which (5.2) is the modified Burgers equation. We apply the entropy condition (5.3) to the N-wave shocks discussed in §2. The condition reads

$$\begin{aligned}(U-H)(U-G)(U+G+H) &\leqslant 0 \quad \text{for} \quad G \leqslant U \leqslant H \\ &\geqslant 0 \quad \text{for} \quad H \leqslant U \leqslant G\end{aligned}$$

or, equivalently,

$$\left.\begin{aligned}H+2G &\geqslant 0 \quad \text{for} \quad H > G \\ &\leqslant 0 \quad \text{for} \quad H < G.\end{aligned}\right\} \tag{5.4}$$

Consider first the head shock. Here $G = 0$, $H = F(X) < 0$, so that $H < G$, $H+2G < 0$ and (5.4) is satisfied, in conformity with our finding, in §2, that there were never any difficulties with

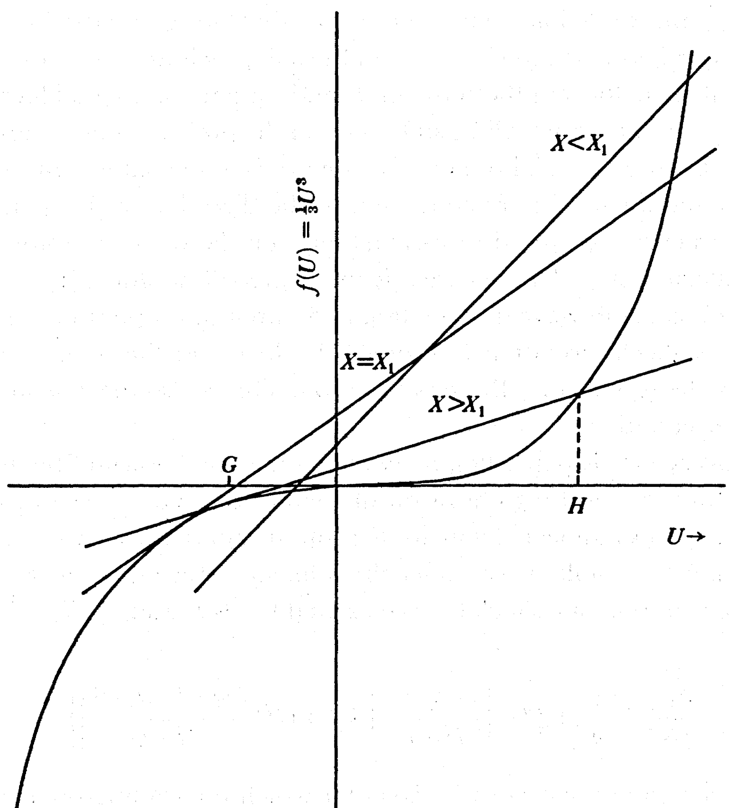

FIGURE 14. Chords of $f(U) = \frac{1}{3}U^3$ for $X \gtreqless X_1$.

either the outer flow or the internal structure in the case of the head shock. Consider second the possibility, alluded to in §2.2, of a shock at $\tau = C_b(X)$ when $X < 1$. For this we would need $0 < G < 1$ and $H = 0$, so that $H < G$, but $H+2G > 0$. Condition (5.4) is thus violated and no such 'back shock' can exist.

Consider finally the tail shock, for which $H > G$, and (5.4) requires $H+2G \geqslant 0$. This is one of the conditions imposed in §2 (and §3) to determine the nature of the lossless flow for $X > X_1$, where it was shown to arise from the need to have a non-singular shock transition from H to G. However, it was simultaneously shown that $H+2G \leqslant 0$ is then also needed for a consistent description of the outer flow, and hence $H+2G = 0$ is required for all $X \geqslant X_1$. Thus the limiting case of the Lax theory applies, i.e. at the lower point $(G, \frac{1}{3}G^3)$ of figure 14 the chord is tangential to the $f(U)$ curve for all $X \geqslant X_1$, or, equivalently, the shock speed $C'(X)$ is equal to the characteristic velocity $G^2 = \frac{1}{4}H^2$ at the side of the shock where the signal is $G(X)$, or, in words, the shock is, from the right in the (X, τ) plane of figure 6, the envelope of the refracted characteristics.

6. Discussion

To emphasize how different the waves governed by cubic nonlinearity can be from those for which the nonlinearity is quadratic, we compare the results derived here for the MBE with those previously obtained for the well-known Burgers equation. The differences are apparent from the outset, because in the cubic case the initial profile of the wave, $V = q(\tau)$, is deformed by stretching in one direction only, whereas the positive and negative parts of the 'quadratic' wave move in opposing directions. Thus the waves that are governed by Burgers equation maintain any antisymmetry present in the initial profile, such as exists in the sine wave case, so that shocks are likely to form at the zero points and, because the signal levels near there will be of equal and opposite strength, will be stationary in the profile. None of this is true for waves governed by the MBE, for which all initial antisymmetry is immediately lost, shocks rarely form at the zero points, and all shocks are convected in the direction of the general deformation. (Note for continuous curves $q(\tau)$ that shocks first form on the wave at positions corresponding to the points of inflexion of $q^{n-1}(\tau)$, where n is the degree of nonlinearity.)

Despite these differences in wave deformation and initial shock position we have shown that the actual process of shock formation ('embryo' shock) under the (cubic) MBE is, to leading order, governed by the (quadratic) Burgers equation. This is also the case for waves governed by the higher-order equations $(n > 3)$.

The major discovery of this paper lies, however, in the breakdown of the usual Taylor type of shock structure and the resulting effects on the outer flow. This breakdown occurs at finite range X_1, when the shocks appear about to develop an internal singularity, and long before they merge back into the whole wave under the influence of dissipation. It is clear, from the general form of the equation of a shock between signal levels H and G $(H > G)$ (equation (2.8) for MBE)

$$\frac{\partial V_0^*}{\partial S^*} = \frac{1}{n}\left\{V_0^{*n} - \left[\frac{H^n - G^n}{H-G}\right] V_0^* + GH\left[\frac{H^{n-1}-G^{n-1}}{H-G}\right]\right\}, \tag{6.1}$$

that this phenomenon can only occur for waves for which the leading-order nonlinearity is at least cubic, i.e. for which there are more than the two roots $V_0^* = H, G$, and hence not waves for which Burgers equation applies. (In fact it can be shown that only equations of odd nonlinearity powers n (i.e. $V^{n-1}V_\tau$) give rise to a shock equation possessing a further (real) root.) This breakdown of the shock structure occurs for waves evolving from a very large class of initial profiles; the sine and N-waves are examples for MBE, for which a certain condition on the signal levels of the outer flow ($H+2G>0$ for MBE) for a singularity-free shock passage is violated. For the general case (n odd) the condition can be interpreted as implying that the ratio of signal levels $\phi = H/G$ must never reach the critical value ϕ_C given by the negative root of the equation

$$\phi^n - n\phi + n - 1 = 0. \tag{6.2}$$

The most significant features of the subsequent wave structure, beyond this breakdown, which is uniquely determined by imposing the condition that the shock be non-singular, can be summarized in the following points:

(i) the signal levels on either side of the shock remain locked in the 'critical' ratio (2:−1 for MBE) with only the overall magnitude of the wave varying;

(ii) information on the outer wave to one side of the shock is no longer passed by characteristics emanating from the initial data, but from characteristics that are refracted through the shock from the other side (or repeatedly refracted through several shocks as in the case of periodic waves), the refracted characteristics all being tangential to the new shock on the right, and;

(iii) the shock tends algebraically to the outer wave at one end, although remaining exponential at the other.

It should be stressed that these features are general to all the waves that produce shocks for which the critical ratio of signal levels is reached, and will remain evident until, after large distances, the effects of dissipation cause the shocks to thicken, drift through the profile, lose their steady-state ('new') form and eventually merge back into the whole wave.

Beyond this point ($X = O(\epsilon^{-2})$ for N-waves, $O(\epsilon^{-1})$ for sinusoidal), dissipation causes the waves to be attenuated so much that nonlinear effects become negligible and, perhaps not surprisingly, the ultimate old-age forms of the sinusoidal and N-waves governed by the MBE are found to be very similar to those of the waves governed by Burgers equation. We have shown that, as in the Burgers case, the wave evolving from the sinusoidal saturates and falls back into the sinusoidal form, although unlike 'Burgers waves' both this and the N-wave old-age form have been swept along from the initial position of the waves.

Before concluding, we note that the asymptotic form of the outer wave, before the final shock breakdown, evolving from the sinusoid, is derived from (3.18), (3.22) and (3.23) for H and ξ independently of the initial profile. Thus, this asymptotic form must hold for all waves evolving from a periodic initial profile which satisfies the conditions $f(\tau+\pi) = -f(\tau)$ (period 2π) for which shocks only develop at intervals of π. Thus any waves evolving from an initial profile of this type will take on the asymptotic form and eventually fall into the saturated sinusoidal mode. The asymptotic form of the N-wave is the exact form for moderate distances and thus, similarly, will represent the final stages of waves evolving from a class of impulse-like disturbances, provided these have vanishing integral over τ. Finally, it may be interesting to consider the wave evolving from a periodic N-wave, more commonly known as the 'sawtooth wave'. Under the MBE, this wave, which is the asymptotic form of a wave evolving from the sinusoidal profile satisfying Burgers equation (see Whitham 1974), will propagate in a way quite different from either the N-wave or the sinusoidal wave.

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Appendix A. Derivation of the modified Burgers equation for nonlinear electromagnetic waves

Consider a plane electromagnetic wave

$$\boldsymbol{E} = (0, E(x,t), 0), \quad \boldsymbol{B} = (0, 0, B(x,t))$$

propagating along the x-axis. Maxwell's equations are

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}, \quad -\frac{\partial H}{\partial x} = \frac{\partial D}{\partial t} \tag{A 1}$$

and the vacuum relation $B = \mu_0 H$ is assumed to hold. The wave propagates in an isotropic dielectric medium in which symmetry demands that the polarization $D - \epsilon_0 E$ be an odd function of E. It is also assumed that all frequencies in the wave are low compared with any resonance frequencies of the medium. Then the relation $D \equiv D(E)$ may be expanded in a power series in E with only odd terms, and in a power series in $\partial/\partial t$ (cf. Gorschkov *et al.* 1974; Landau *et al.* 1960, ch. 12 and 13); thus

$$D = \epsilon E + \alpha E^3 + \ldots - \beta \frac{\partial E}{\partial t} - \gamma \frac{\partial^2 E}{\partial t^2} - \ldots, \tag{A 2}$$

in which α may have either sign, as may γ, whereas β represents a damping coefficient and is non-negative. Taking just the terms explicitly quoted in (A 2), the system (A 1) takes the form (with $c^2 = 1/\epsilon\mu_0$)

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \alpha\mu_0 \frac{\partial^2 E^3}{\partial t^2} + \beta\mu_0 \frac{\partial^3 E}{\partial t^3} + \gamma\mu_0 \frac{\partial^4 E}{\partial t^4} = 0,$$

in which the last three terms are assumed locally small compared with the first two. Confining attention to waves propagating in the positive x-direction, we can replace $\partial^2/\partial x^2 - (1/c^2)\,\partial^2/\partial t^2$ by $(-2/c)\,(\partial/\partial t)\,(\partial/\partial x + (1/c)\,\partial/\partial t)$, approximately, and then integrate with respect to t. This gives

$$\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right) E + \tfrac{1}{2}(\alpha\mu_0 c)\,\frac{\partial E^3}{\partial t} - \tfrac{1}{2}(\beta\mu_0 c)\,\frac{\partial^2 E}{\partial t^2} - \tfrac{1}{2}(\gamma\mu_0 c)\,\frac{\partial^3 E}{\partial t^3} = 0,$$

or

$$\frac{\partial E}{\partial x} + (\tfrac{3}{2}\alpha\mu_0 c)\, E^2 \frac{\partial E}{\partial t'} = (\tfrac{1}{2}\beta\mu_0 c)\,\frac{\partial^2 E}{\partial t'^2} + (\tfrac{1}{2}\gamma\mu_0 c)\,\frac{\partial^3 E}{\partial t'^3} \tag{A 3}$$

if x and $t' = t - x/c$ are taken as independent variables. This is the modified Burgers–Korteweg de Vries equation, reducing to the modified Burgers equation if the dispersive term with coefficient γ is dropped. The natural boundary condition for (A 3) is a signalling one, in which $E(x=0,\ t) \equiv E(x=0,\ t')$ is prescribed, and E sought for $x > 0$, all t'.

The above derivation can readily be put on a formal basis as, for example, in Leibovich & Seebass (1974, ch. 4). Formal multiple-scales methods were used by Teymur & Suhubi (1978) and Lee-Bapty (1981) to derive the modified Burgers equation for transverse waves in a viscoelastic solid.

Now define dimensionless variables according to

$$E = E_0 V, \quad X = \tfrac{3}{2}\alpha\mu_0 c^2 E_0^2 (\omega x/c), \quad \tau = \omega t', \tag{A 4}$$

where E_0 is a typical magnitude of $E(x = 0, t)$, ω a typical frequency, and drop the dispersive term. Then (A 3) gives precisely (2.1), with the identification

$$\epsilon = \beta\omega/3\alpha E_0^2. \tag{A 5}$$

Appendix B. Higher-order terms

The motivation for considering the higher-order effects generated by the new type of shock, in which the signal levels are in the ratio $2:-1$, is to consolidate the leading-order theory and to determine the corrections due to diffusivity to the shock location, i.e. $S_{t_0}^*(X)$ in the N-wave problem and $S_0^*(X)$ in the sinusoidal one. It is necessary to know these functions to make a complete study of the breakdown of the shock structure for large X (see condition (ii), §2.6). Apart from determining these functions, the higher-order analysis reveals some interesting features of the new shock, such as the need for a new local 'intermediate' region about $V = -\frac{1}{2}H(X)$ to enable successful matching from the shock to the lossless wave on the right. It should be noted, however, that the introduction of this region also generates effects that we find it difficult to explain completely. For this reason, and also because many of the expressions involved are very long and complicated, we shall give only a brief outline of the issues. Details can be found in Lee-Bapty (1981).

Consider the general case of a shock of the new type, with signal level $H(X)$ on the left and $-\frac{1}{2}H(X)$ on the right, positioned at $\tau = C(X)$. Formal matching is carried out from left to right (in τ) across the shock at a point $X = \xi(> X_1)$; see figures 6 and 8. Matching the lossless and shock solutions on the left implies lossless and shock expansions

$$V(X, S < 0) = V_0 + \epsilon V_1 + o(\epsilon), \tag{B 1a}$$

$$V(X, S^*) = V_0^* + \epsilon V_1^* + o(\epsilon), \tag{B 1b}$$

for suitable V_0, V_1, V_0^*, V_1^*. Matching the shock solution to the lossless solution on the right fails beyond leading order. A new 'intermediate' region is needed around $V = -\frac{1}{2}H(X)$ in which an approximation to $\partial V/\partial X$ must balance the nonlinear and diffusive effects dominant in the 'inner shock'. This is achieved by the scalings $\bar{S} = S/\epsilon^{\frac{2}{3}} = \epsilon^{\frac{1}{3}} S^*$, $V = -\frac{1}{2}H(X) + \epsilon^{\frac{1}{3}}\bar{V}_1 + o(\epsilon^{\frac{1}{3}})$, which give

$$\frac{\partial^2 \bar{V}_1}{\partial \bar{S}^2} + H(X)\bar{V}_1\frac{\partial \bar{V}_1}{\partial \bar{S}} = -\tfrac{1}{2}H'(X) \tag{B 2}$$

and a solution in terms of Airy functions. Matching to the shock to the left and to the lossless solution to the right can now be carried out.

The intermediate region and solution now dictate the presence of $O(\epsilon^{\frac{2}{3}})$ terms in both the shock expansion and the lossless expansion on both sides of the shock (in addition to the 'obvious' terms already quoted in (B 1)). Logarithmic terms are also forced into the expansions at $O(\epsilon)$, and one has to assume the developments

$$V(X, S) = V_0 + \epsilon^{\frac{2}{3}} V_{01} + \epsilon V_1 + \epsilon \ln \epsilon \, V_2 + o(\epsilon), \tag{B 3a}$$

$$V(X, S^*) = V_0^* + \epsilon^{\frac{2}{3}} V_{01}^* + \epsilon V_1^* + \epsilon \ln \epsilon \, V_2^* + o(\epsilon), \tag{B 3b}$$

$$V(X, \bar{S}) = \bar{V}_0 + \epsilon^{\frac{1}{3}} \bar{V}_1 + \epsilon^{\frac{2}{3}} \bar{V}_2 + \epsilon^{\frac{2}{3}} \ln \epsilon \, \bar{V}_3 + o(\epsilon^{\frac{2}{3}}), \tag{B 3c}$$

in the lossless, shock and intermediate regions, respectively. Solutions for all the coefficient functions are given in Lee-Bapty (1981).

Now consider the sinusoidal problem of §3, where the shocks occur at intervals of π and where the solution at one shock is related to that at the previous one by $V(\tau+\pi) = -V(\tau)$. Because of this repetition the lossless solution along a characteristic emanating from the right of a shock at $X = \xi_{n-1}$ (point A in figures 9 and 15a) should equate to that along the characteristic running into the left of the shock of the next half-cycle at $X = \xi_n$ (point B in figures 9 and 15a) to all orders in ϵ.

When the matching is combined with this requirement, the large X asymptotics of the shock displacement due to diffusivity can be found (with the aid of (3.24)), namely

$$S_0^*(X) \sim S_{00} X \ln X + S_{01} X, \tag{B 4}$$

where the S_{ij} are complicated purely numerical constants. This substantiates the claim of §3 that a global non-uniformity (with violation of all the assumptions of weak shock theory) takes place at ranges $X = O(\epsilon^{-1})$. An important discrepancy, however, is that the equivalence of the lossless solutions at the points A and B mentioned earlier is not maintained in respect of the terms $O(\epsilon^{\frac{2}{3}})$ and $O(\epsilon \ln \epsilon)$ (see later in this section).

For the N-wave problem a different strategy is called for, in which second-order matching of the tail shock to the lossless flow on its left is first employed to yield $S_{t_0}^*(X)$, following which the head-shock displacement $S_{h_0}^*(X)$ is to be found from second-order matching of the head shock to the lossless flow on its left. This strategy fails, however, because of indeterminacy in the lossless solution, which is not uniformly valid in the initial development near $\tau = -1$ and $X = 0$. In fact, with the scalings $V = O(1)$, $X = O(\epsilon)$, $\tau + 1 = O(\epsilon)$ the complete modified Burgers equation holds. Inability to solve the MBE in this initial region leads to indeterminacy of parts of the lossless solution, and inability to determine the shock displacements.

We have already noted one difficulty associated with the $O(\epsilon^{\frac{2}{3}})$ and $O(\epsilon \ln \epsilon)$ terms forced into the lossless expansions by the new type of shock (with its exponential decay on the left and algebraic decay on the right, necessitating the introduction on the right of the intermediate region as a buffer between the shock and lossless regions). Another difficulty is that these terms cannot in general be present for $X < X_1$, for insertion of the lossless expansion (B 3a) into the modified Burgers equation with an arbitrary smooth condition at $X = 0$ shows that the expansion must proceed with integral powers of ϵ only. How then do the $O(\epsilon^{\frac{2}{3}})$ and $O(\epsilon \ln \epsilon)$ terms get switched on? One possibility is that when the initial condition is not smooth they are switched on by the 'boundary' or 'initial' layer, which certainly exists for the N-wave. That possibility is not open for a smooth initial condition, and the most plausible alternative seems to be to suggest that the terms are switched on only when a shock has assumed its new form at $X = X_1$, with subsequent switching on at $X = X_n$ ($n \geqslant 2$ for matching between the shocks of the sinusoidal problem, $n = 2$ for matching between the tail and head shocks of the N-wave).

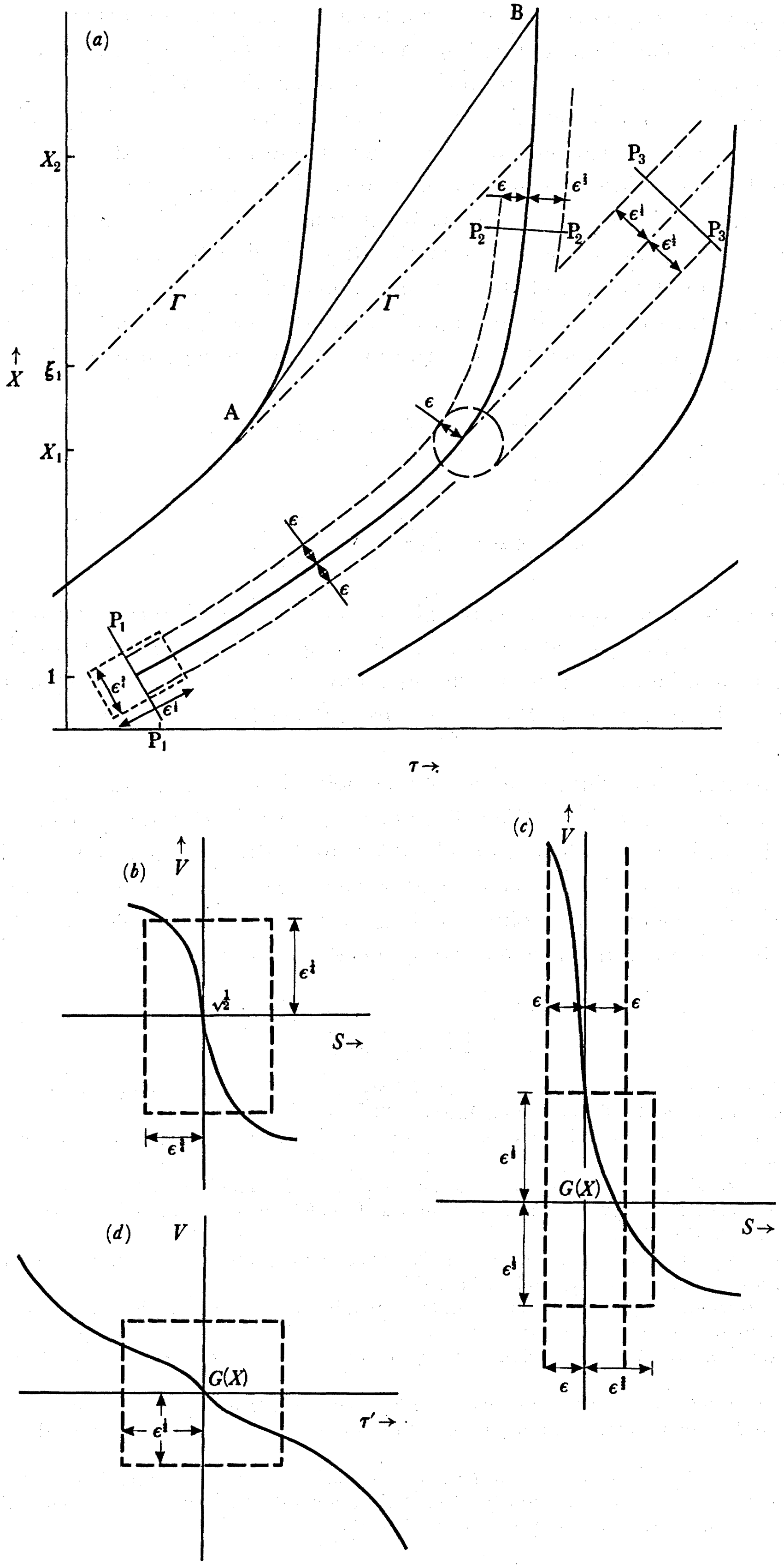

FIGURE 15. (*a*) Scaling diagram in the (τ, X) plane: –·–, tangential characteristic curves Γ; ------, edges of shock region and of other regions in which dissipative effects are significant; ——, shock paths. The points A and B are identified in the text, as are ξ_1, X_1, X_2. (*b*, *c*, *d*) Sketches of profiles $V(S)$ or $V(\tau')$ at sections P_1P_1, P_2P_2, P_3P_3 of figure 15*a*.

This would imply the existence of some transitional behaviour in horizontal bands about $X = X_n$ in the (τ, X) plane, but the scalings and structure here elude us at present.

Finally we remark that the transition from a classical type of shock to the new kind is governed by the full MBE around $X = X_1$. The scalings are $V = O(1)$, $X - X_1 = O(\epsilon)$, $\tau - C(X_1) = O(\epsilon)$; see figure 15*a–d*. From this region the horizontal bands across which certain terms of the lossless expansion are discontinuous must presumably emanate. At X_1, there is also a discontinuity in $\mathrm{d}G/\mathrm{d}X$, and a consequent discontinuity in $\partial V_0/\partial X$ across the first tangential characteristic Γ. This characteristic marks the dividing line between information coming directly from the initial data, and that passing along refracted characteristics through the shock. To alleviate the discontinuity in $\partial V_0/\partial X$ there must be significantly greater dissipative activity in a narrow region around Γ and running the whole length of Γ. This is expressed analytically by the scalings

$$\tau' = \epsilon^{\frac{1}{2}}\tilde{\tau}, \quad V = G(X_1) + \epsilon^{\frac{1}{2}}\,\tilde{V}_1 + O(\epsilon)$$

and the ordinary Burgers equation

$$\frac{\partial \tilde{V}_1}{\partial X} + 2G(X_1)\,\tilde{V}_1 \frac{\partial \tilde{V}_1}{\partial \tilde{\tau}} = \frac{\partial^2 \tilde{V}_1}{\partial \tilde{\tau}^2}, \tag{B 5}$$

$\tau'(\tau, X) = 0$ being the equation of Γ. Although the matching conditions above and below Γ are known, together with the general solution for $\tilde{V}_1$, a unique solution for $\tilde{V}_1$ cannot be found because of insufficient knowledge of the full MBE region at X_1. None the less, (B 5) is interesting as showing how the new type of shock leads to enhanced dissipative activity far from the shocks.

Failure to resolve these difficulties associated with higher-order terms is by no means uncommon in nonlinear problems. We believe this in no way undermines the credibility of our proposals for the leading-order structure, which are internally consistent, and substantiated by the numerical results of §4. We can see no way of improving upon the higher-order description sketched here until general exact solutions of the MBE become available, and of this there is absolutely no hint in any current work in nonlinear-wave theory.

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